Test 1   Ch 12-14   Review sheet posted
Practice test 1 (5%+)

Math 2433 Week 3

14.1 Examples of real-valued functions of two and three variables

Finding domain and range:

$f(x) = y$
domain $= \cancel{x}$ input
range $= \cancel{y}$ output

1. $f(x, y) = \sqrt{xy}$

$\text{dom}(f) = \{(x,y) \mid xy \geq 0\}$

$(x,y) \mid$ the pt (x,y) is in QI or QIII or on axis

range $[0, \infty)$ non-negative reals

pos real #'s : $(0, \infty)$

2. $f(x, y) = \dfrac{1}{\sqrt{x-y}}$

$\text{dom}(f) = \{(x,y) \mid x > y\}$

range: $(0, \infty)$

3. $f(x,y,z) = \dfrac{\sqrt{z}}{x^2 - y^2}$

$x^2 \neq y^2$

$x \neq y \quad x \neq -y \quad |x| \neq |y|$

$\text{dom}(f) = \{(x,y,z) \mid z \geq 0,\ |x| \neq |y|\}$

range: $(-\infty, \infty)$

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biggest value

smallest

2 3 0

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1. Give the domain of $f(x,y) = \sqrt{4-x^2} - \sqrt{9-y^2}$
   a. $\{(x,y) \mid -2 \leq x, -3 \leq y\}$
   b. $\{(x,y) \mid x \leq 2, y \leq 3\}$
   c. $\{(x,y) \mid -2 < x < 2, -3 < y < 3\}$
   d. $\{(x,y) \mid -2 \leq x \leq 2, -3 \leq y \leq 3\}$
   e. None of these

Range: $[-3, 2]$

14.2 The Quadric Surfaces and Projections
Watch the week 3 video on Surfaces before class!!!

The ***quadric surfaces*** are surfaces that can be written in the form:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Hx + Iy + Jz + K = 0$$

where $A, B, C, \ldots, K$ are constants.

There are nine distinct types (**see book for all 9 and examples**). These are important, so get familiar with them.

Projections:

Suppose you have two surfaces that intersect in a space curve C:

The curve C is the set of all (x, y, z) such that $z = f(x, y)$ and $z = g(x, y)$ and so the set of all points (x, y, z) such that $f(x, y) = g(x, y)$ create a vertical cylinder through C. If we find all the points $(x, y, 0)$ where $f(x, y) = g(x, y)$ then we have a *projection* of C onto the *xy-plane*.

intersection in terms of x+y only

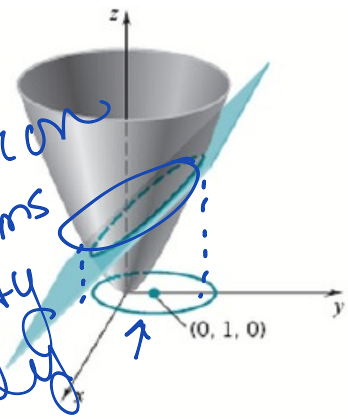


Figure 14.2.11

Example:

A sphere $x^2 + y^2 + (z-1)^2 = \frac{3}{2}$ and the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect in a space curve C. Determine the projection of C onto the xy-plane.

$x^2 + y^2 = (1 + z^2)$

$1 + z^2 + (z-1)^2 = 3/2$

$1 + z^2 + z^2 - 2z + 1 = 3/2$

$2z^2 - 2z + 1/2 = 0$

$4z^2 - 4z + 1 = 0$

$(2z-1)^2 = 1$

$z = 1/2$;

$x^2 + y^2 + (-1/2)^2 = 3/2$

$x^2 + y^2 + 1/4 = 6/4$

$x^2 + y^2 = 5/4$

$x^2 + y^2 - (1/2)^2 = 1$

$x^2 + y^2 - 1/4 = 1$

$\boxed{x^2 + y^2 = 5/4}$

circle

Another example: Determine the projection onto the *xy*-plane: $y^2+z-4=0$ & $x^2+3y^2=z$

$$z=(4-y^2)$$

$$x^2+3y^2=4-y^2$$

$$x^2+4y^2=4 \quad \text{or} \quad \frac{x^2}{4}+y^2=1$$

ellipse

Popper 3

2. The surfaces intersect in a space curve C. Determine the projection of C onto the *xy*-plane for the sphere $x^2+y^2+(z-2)^2=2$ and the cone $x^2+y^2=z^2$.
   a. $x^2-y^2=1$
   b. $x^2+y^2=2$
   c. $x^2+y^2=1$
   d. $x^2-y^2=2$
   e. None of these

14.3 Level Curves and Level Surfaces

Look over book examples!!!

When we talk about the graph of a function with two variables defined on a subset D of the xy-plane, we mean: $z = f(x, y) \quad (x, y) \in D$

If c is a value in the range of f then we can sketch the curve $f(x,y) = c$. This is called a **level curve**.

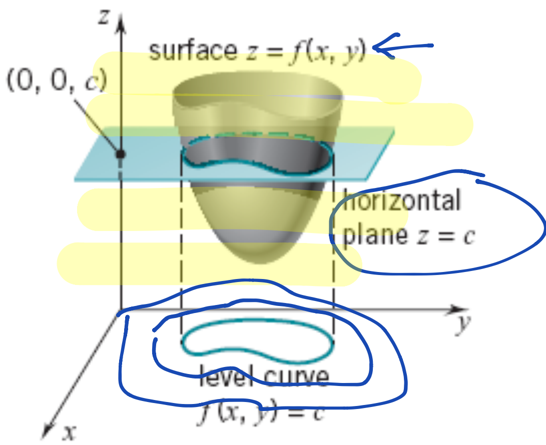


Figure 14.3.4

$z = \text{constant}$

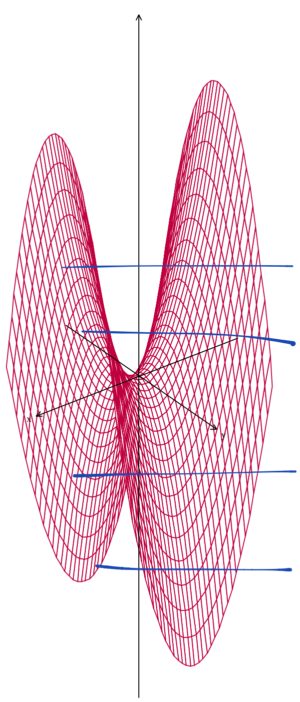
A collection of level curves can give a good representation of the 3-d graph.

$x^2 - y^2 = c \Rightarrow$ hyperbola

Examples:
Identify the level curves $f(x, y) = c$ and sketch the curves corresponding to the indicated values of c.

1. $f(x, y) = x^2 - y^2$

$c = -2, -1, 0, 1, 2$



$x^2 - y^2 = -2$

$y^2 - x^2 = 2$

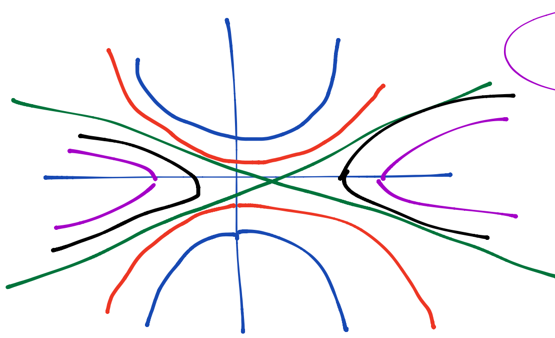
$\frac{y^2}{2} - \frac{x^2}{2} = 1$

$x^2 - y^2 = -1$

$y^2 - x^2 = 1$

$x^2 - y^2 = 0$

$x = \pm y$



$x^2 - y^2 = 1$

$x^2 - y^2 = 2$

$e^{\ln(\text{thing})} = \text{thing}$

$z = c$

2. Identify the level curves:

   a. $f(x,y) = \ln(x - y^2)$ $= z$

$c = \ln(x - y^2)$

$e^c = x - y^2 \Rightarrow$ parabola

still const

   b. $f(x,y) = e^{(x^2+2y^2)}$

$c = e^{(x^2+2y^2)}$

$\ln(c) = x^2 + 2y^2$

ellipse

3. Identify the c-level surface and sketch it.

$f(x,y,z) = 4x^2 - 9y^2 - 72z \qquad c = 0$

no longer $z = c$

its $f(x,y,z) = c$

$4x^2 - 9y^2 - 72z = c$

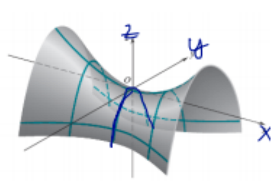
$4x^2 - 9y^2 - 72z = 0$

$4x^2 - 9y^2 = 72z$

$\frac{x^2}{18} - \frac{y^2}{8} = z$

The hyperbolic paraboloid: symmetry about the xz-plane and yz plane. Section xy-plane are hyperbolas; sections parallel to the other coordinate planes are parab

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$



hyperbolic paraboloid

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3. Identify the level curves of the given surface.

$$f(x,y)=\frac{1}{x-y^2}$$

$$\frac{1}{x-y^2}=C$$

a. parabolas
b. hyperbolas
c. ellipses
d. exponentials
e. circles
f. none of these

Partial Derivatives (parts from 14.4 and 14.6)

The *partial derivative of f with respect to x* is the function *fx* obtained by differentiating *f* with respect to *x*, treating *y* as a constant.

The *partial derivative of f with respect to y* is the function *fy* obtained by differentiating *f* with respect to *y*, treating *x* as a constant.

Examples:

$$\frac{d}{dx}(cx)=c$$

Find the partial derivatives:

1. $f(x,y)=3x^2-2y+xy$

$$f_x(x,y)=\frac{\partial f}{\partial x}=6x-0+y=6x+y$$

$$f_y(x,y)=\frac{\partial f}{\partial y}=0-2+x=x-2$$

2. $f(x,y) = e^{x+y^2} + \ln\left(\frac{x}{x^2+y}\right) = e^{x+y^2} + \ln(x) - \ln(x^2+y)$

$$f_x(x,y) = e^{x+y^2} + \frac{1}{x} - \frac{2x}{x^2+y}$$

$$f_y(x,y) = 2ye^{x+y^2} - \frac{1}{x^2+y}$$

3. $f(x,y,z) = 3x^2z - e^y\sqrt{z} + \sqrt{x^2+y^2}$

$$f_x(x,y,z) = 6xz + \frac{1}{2}(x^2+y^2)^{-1/2}(2x)$$

$$= 6xz + \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y(x,y,z) = -e^y\sqrt{z} + \frac{1}{2}(x^2+y^2)^{-1/2}(2y)$$

$$= \frac{y}{\sqrt{x^2+y^2}} - e^y\sqrt{z}$$

$$f_z(x,y,z) = 3x^2 - e^y\,\frac{1}{2}z^{-1/2} + 0$$

$$= 3x^2 - \frac{e^y}{2\sqrt{z}}$$

Higher order derivatives:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2},$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x},$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y},$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}.$$

What does this mean?

Example:

$$f(x, y) = 2x^2 \cos(y) + 3y^2 \sin(x)$$

$$\boxed{f_x = 4x\cos(y) + 3y^2\cos(x)} \qquad f_y = -2x^2\sin(y) + 6y\sin(x)$$

$$f_{xx} = 4\cos(y) - 3y^2\sin(x) \qquad f_{yy} = -2x^2\cos(y) + 6\sin(x)$$

$$f_{xy} = -4x\sin(y) + 6y\cos(x) \qquad f_{yx} = -4x\sin(y) + 6y\cos(x)$$

$$f_{xy} = f_{yx}$$

In the case of a function of three variables you can look for three first partials f_x, f_y, f_z and there are NINE second partials: $f_{xy}, f_{yx}, f_{xz}, f_{zx}, f_{yz}, f_{zy}, f_{xx}, f_{yy}, f_{zz}$

Examples:

1. $f(x,y,z) = 4xe^y + 3ye^z + 2ze^x$ $= 4xe^y + 3ye^z + 2ze^x$

$f_x = 4e^y + 2ze^x$	$f_y = 4xe^y + 3e^z$	$f_z = 3ye^z + 2e^x$
$f_{xx} = 2ze^x$	$f_{yy} = 4xe^y$	$f_{zz} = 3ye^z$
$f_{xy} = 4e^y$	$f_{yx} = 4e^y$	$f_{zx} = 2e^x$
$f_{xz} = 2e^x$	$f_{yz} = 3e^z$	$f_{zy} = 3e^z$

2. $f(x,y,z) = x^2y + y^2z + z^2x$

$f_x = 2xy + z^2$ | $f_y = x^2 + 2yz$ | $f_z = y^2 + 2zx$

$f_{xx} = 2y$ | $f_{yy} = 2z$ | $f_{zz} = 2x$

$f_{xy} = 2x$ | $f_{yx} = 2x$ | $f_{zx} = 2z$

$f_{xz} = 2z$ | $f_{yz} = 2y$ | $f_{zy} = 2y$

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4. If $f(x,y) = e^{\sin x} + x^5y + \ln(1+y^2)$, then $f_{yx} =$

   a. $5x^4$

   b. $\cos x\, e^{\sin x}$

   c. $\cos x\, e^{\sin x} + 5x^4$

   d. $5x^4y + \frac{2y}{1+y^2}$

   e. $x^5 + \frac{2y}{1+y^2}$

   f. None of these

f_{yx}

find f_y 1st then

$(f_y)_x$

5. Find the curvature at $t = 0$ of the curve described by $x = 2 - \cos t, \quad y = \sqrt{2}\sin t, \quad z = \cos t$

a. 0
b. 1/3
c. 1/2
d. $1/\sqrt{2}$
e. $\sqrt{2}$
f. None of these

$$K = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$