Math 3339

Section 27204
MWF 10-11:00am AAAud 2

Bekki George
bekki@math.uh.edu
639 PGH

Office Hours:
M & Th noon – 1:00 pm & T 1:00 – 2:00 pm
and by appointment

Popper 04

Empirical probability (experimental probability) of an event is the ratio of the number of outcomes in which a specified event occurs to the total number of trials, not in a theoretical sample space but in an actual experiment.

1. A die is rolled 60 times with the following results recorded:

Outcome	1	2	3	4	5	6
Frequency	10	6	12	9	8	15

The empirical probability of getting a 3 is:

$\frac{12}{60}$

A. 1/12
B. 1/10
C. 1/6
D. 1/5
E. 1/4

Expected Values

Consider the table below which gives the number of years required to obtain a Bachelor's degree for graduates of high school A, and the number of students who needed each:

Years X	3	4	5	6
Number of Students	17	23	38	19
$f(x)$	$17/97$	$23/97$	$38/97$	$19/97$

97 students

How would compute the "average" number of years required by graduates of high school A?

$$E[X] = \frac{3(17) + 4(23) + 5(38) + 6(19)}{97} \approx 4.61$$

$$= 3\left(\frac{17}{97}\right) + 4\left(\frac{23}{97}\right) + 5\left(\frac{38}{97}\right) + 6\left(\frac{19}{97}\right)$$

probabilities

$$E[X] = \sum_i x_i f(x_i)$$

The "average" value of a rv X is called the "expected value" of X.

Def: Let X be a discrete rv with set of possible values D and pmf p. The ***expected value*** or ***mean value*** of X, denoted E[X] or μ_X or just μ is $\boxed{E[X] = \sum_{x \in D} x \cdot f(x)}$

Properties of Expected Value and Variance:

$$\frac{5+5+5}{3} = 5$$

1. $E[c] = c$ for any constant $c \in \mathbb{R}$
2. $E[aX + bY] = aE[X] + bE[Y]$
3. $E[h(X)] = \sum_{x \in D} h(x) f(x)$
4. $V(X) = E\left[(X-\mu)^2\right]$ or $\boxed{V(X) = E[X^2] - E[X]^2}$
5. $V(aX+b) = a^2 V(X)$

variance

$E[a \cdot X] = a \cdot E[X]$

$E[X+b] = E[X] + b$

$E[aX \pm b] = aE[X] \pm b$

Ex: Let X have pmf. given by

x^2	1	4	9	16
x	1	2	3	4
$f(x)$	0.4	0.2	0.3	0.1

Determine $E[X]$, $E[X^2]$, and use the formula $\sigma^2 = E[X^2] - (E[X])^2$ to determine the standard deviation of X.

$$E[X] = 1(.4) + 2(.2) + 3(.3) + 4(.1) = 2.1$$

$$E[X^2] = 1(.4) + 4(.2) + 9(.3) + 16(.1) = 5.5$$

$$V[X] = \sigma_x^2 = 5.5 - 2.1^2 = 1.09$$

$$\sigma = \sqrt{1.09} = 1.044$$

Determine the expected value and variance of the rv Y defined by $Y = 5X - 1$, where X is given in the previous problem.

$$E[Y] = E[5X-1] = 5E[X] - 1$$
$$= 5(2.1) - 1$$
$$= 10.5 - 1 = 9.5$$

$$V[Y] = V[5X-1] = V[5X] = 5^2 V[X] = 25(1.09)$$
$$= 27.25$$

Popper 04

Suppose X is a rv with $E[X] = 3.7$ and $Var[X] = 2.25$ and $Y = 2X - 3$. Find:

2. $E[Y]$

   a. 7.4

   b. 4.4

   c. 3.7

   d. 2.5

   e. none of these

3. $Var[Y]$

   a. 9

   b. 4.5

   c. 3

   d. 3.7

   e. none of these

The Binomial Probability Distribution

Definition: A *Bernoulli Trial* is a random experiment with the following characteristics:

$X = 0$ or $X = 1$

1. The outcome can be classified as either "success" or "failure" (where these are mutually exclusive and exhaustive).
2. The probability of success is p, so the probability of failure is $q = 1 - p$.

e.g. a coin is flipped (heads or tails), someone is pulled over for speeding (ticket or warning), etc.

$p = \frac{1}{2}$ $q = \frac{1}{2}$

Dice where success is rolling 2

$p = 1/6$ $q = 5/6$

Suppose that a coin is flipped. Let X be the random variable that indicates that heads was flipped (i.e. $X = 1$ if heads $X = 0$ if tails). Here heads represents "success" and tails represents "failure" so that X is a Bernoulli random variable.

$\frac{1}{2}$ $\frac{1}{2}$

Suppose that we flip a coin 10 times. This is a *sequence* of Bernoulli trials. We are interested in calculating the probability of obtaining a certain number of heads. Let X_i indicate heads on the i-th flip.

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0 or 1

Define $Y = X_1 + X_2 + ... + X_{10}$. What does Y represent? # of heads in 10 flips

What is the probability that $Y = 0$? $P(Y=0) = P(\text{no heads}) = \frac{1}{2^{10}} = \frac{1}{1024}$

What is the probability that $Y = 1$? $P(Y=1) = P(\text{one head}) = \frac{10}{2^{10}}$

What is the probability that $Y = 2$? $P(Y=2) = \frac{{}_{10}C_2}{2^{10}} = {}_{10}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^8$

What is the probability that $Y = n$ where $n = 0,1,2,...,10$? $P(Y=n) = \frac{{}_{10}C_n}{2^{10}}$

Here Y is the sum of 10 independent Bernoulli trials. We call this type of random variable a *Binomial* random variable.

A random variable X is a *Binomial* random variable if the following conditions are satisfied:

1. X represents the number of successes on n Bernoulli trials. fixed # of trials
2. The probability of success for each trial is p. Same p for each trial
3. The trials are mutually independent.

If X is a binomial random variable with probability p of success on each of n trials, we write $X \sim \text{Binomial}(n, p)$

n = # of trials

x = # of successes

If $X \sim \text{Binomial}(n, p)$, then

$$P(X = x) = \underbrace{\binom{n}{x}}_{{}_nC_x} p^x (1-p)^{n-x}, \quad \text{where } x = 0, 1, 2, \ldots, n$$

R commands:

_binom( )

$$P(X = x) = dbinom(x, n, p)$$

$$P(X \le x) = pbinom(x, n, p) = P(X=0) + P(X=1) + P(X=2) + \cdots + P(X=x)$$

$$P(X > x) = 1 - pbinom(x, n, p)$$

If $p = \frac{X}{n}$, what is the expected value of this statistic?

$$\boxed{\begin{aligned} E[X] &= np \\ \sigma^2 &= np(1-p) \end{aligned}}$$

$\sigma = \sqrt{np(1-p)}$

Example: Suppose that at a 4-way stop in a certain subdivision, only 12% of drivers come to a complete stop. What is the probability that among $\underset{n}{\underline{8}}$ drivers, at least 6 of them will run the stop sign?

$\underbrace{0\ 1\ 2\ 3\ 4\ 5}\ \boxed{6\ 7\ 8}$ $p = .88$

dbinom(6, 8, .88) + dbinom(7, 8, .88) + dbinom(8, 8, .88)

$P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8)$

$1 - P(X \leq 5) = 1 - \text{pbinom}(5, 8, .88) = .939$

What is the expected number of drivers who will run the stop sign?

$8(.88) = 7.04$

$\sim \text{Binomial}(n, p)$

> 5

Ex: Suppose $X \sim \text{Binomial}(12, 0.3)$.

Determine $P(2 \le X < 5)$

$$\boxed{0 \quad 1 \quad 2 \quad 3 \quad 4} \quad 5 \quad \overbrace{6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12}^{>5}$$

$= P(X=2) + P(X=3) + P(X=4)$ = dbinom(2,12,.3) + dbinom(3,12,.3) + dbinom(4,12,.3)

$= P(X \le 4) - P(X \le 1)$ = pbinom(4,12,.3) - pbinom(1,12,.3)

=
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> pbinom(4,12,.3)-pbinom(1,12,.3)
[1] 0.6386304
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Determine $P(X > 5)$

$1 - P(X \le 5)$

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> 1-pbinom(5,12,.3)
[1] 0.1178487
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Popper 04

$p = .7$

4. Seventy percent of all trucks undergoing brake inspection at a certain facility pass the inspection. Consider a group of 15 trucks. What is the probability that between 10 and 12 trucks inclusively pass the inspection?

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$P(10 \le X \le 12)$

$= P(X \le 12) - P(X \le 9)$

a. 0.5948
b. 0.3887
c. 0.2186
d. 0.5008
e. none of these

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> pbinom(12,15,.7)-pbinom(9,15,.7)
[1] 0.5947937
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5. Find the mean and standard deviation for the rv *X* as described in the problem above.

a. Mean = 10.5, standard deviation = 3.15
b. Mean = 4.5, standard deviation = 3.15
c. Mean = 10.5, standard deviation = 1.77
d. Mean = 4.5, standard deviation = 1.77
e. None of these

$15(.7) =$

$\sigma = \sqrt{15(.7)(.3)}$

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> 15*.7
[1] 10.5
> sqrt(15*.7*.3)
[1] 1.774824
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