Math 3339

Section 27204
MWF 10-11:00am AAAud 2

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Office Hours:
M & Th noon – 1:00 pm & T 1:00 – 2:00 pm
and by appointment

Popper 11

1.Suppose X is a continuous rv with $F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^3}{27} & 0 \le x < 3 \\ 1 & x \ge 3 \end{cases}$

Find $P(2 < X < 3)$

a. 8/27

b. 19/27

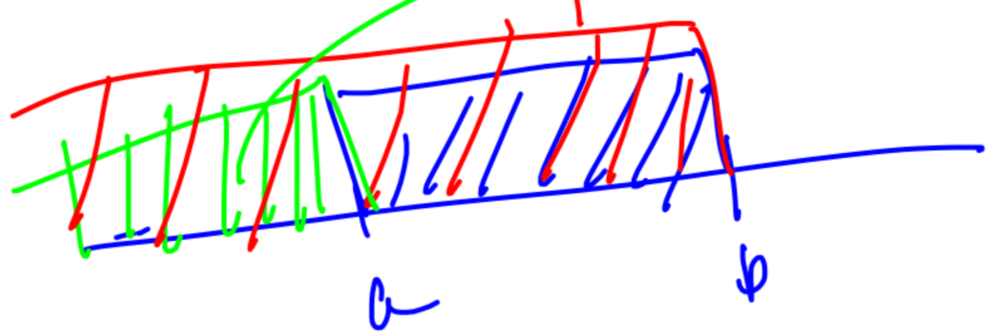
c. 2/3

d. 26/27

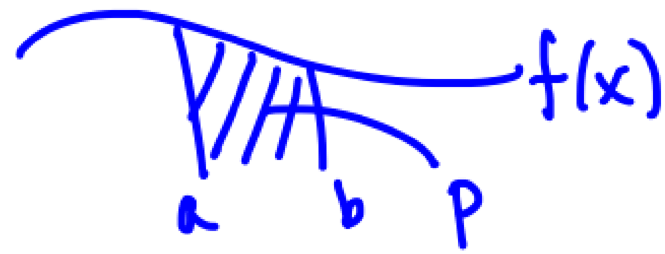
e. none of these

$P(X \le x) = P(X < x)$

$P(a < X < b) = F(b) - F(a)$



$$F'(x) = f(x)$$



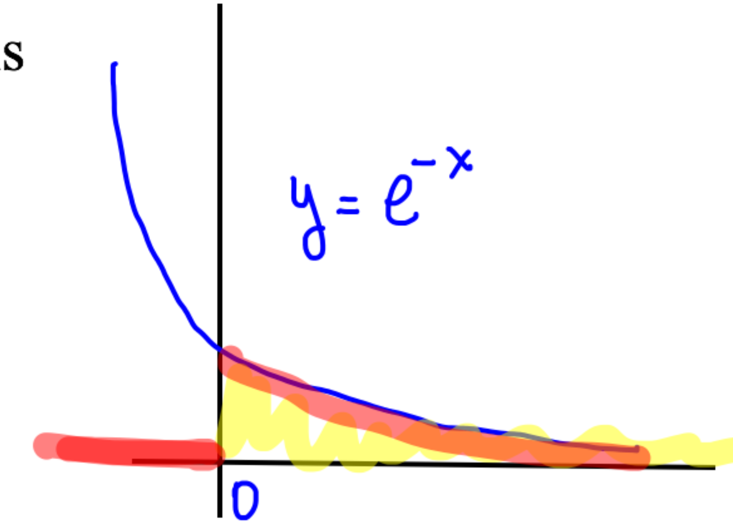
$$F(x) = P(X \leq x)$$

Given $f(x)$ + want $F(x) \rightarrow$ integrate

$$E[x] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

The Exponential Distribution

Def. A random variable X has an *exponential distribution* with rate parameter $\lambda > 0$, if its cumulative distribution is



$P(X < x)$

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mu = \frac{1}{\lambda}$$ so $\lambda = \frac{1}{\mu}$

$$\sigma^2 = \frac{1}{\lambda^2}$$

R command pexp(λ) or dexp(λ) for probabilities and qexp(*quantile*, λ) for quantiles.

$P(X < c) =$ quantile

Ex: Suppose the time a child spends waiting at for the bus as a school bus stop is exponentially distributed with mean 3 minutes. Determine the probability that the child must wait at least 5 minutes on the bus on a given morning.

μ

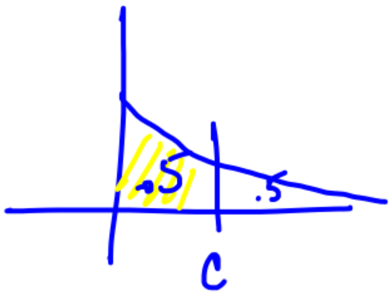
$\lambda = 1/3$ $\quad P(X \geq 5) = \underbrace{1 - P(X \leq 5)} = 1 - F(5)$

$= 1 - (1 - e^{-\frac{1}{3}(5)})$

$= .1889$

$1 - \text{pexp}(5, 1/3)$

$= .1889$



What is the median wait time?

$P(X \leq c) = .5$

$F(c) = .5$

$1 - e^{-\frac{1}{3}c} = .5$

$.5 = e^{-\frac{1}{3}c}$

$\ln(.5) = -1/3 \cdot c$

$c = 2.079$

$\text{qexp}(.5, 1/3)$

The Gamma Distribution

We say that X has a *Gamma Distribution* with parameters $\alpha > 0$ and $\beta > 0$ if X has p.d.f.

$$\rightarrow f(y) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{-y/\beta}, \; 0 \le y < \infty$$

Note: Here $\Gamma(\alpha) = \int_0^{\infty} w^{\alpha-1} e^{-w}\, dw$ (The Gamma Function)

$\Gamma(\textit{something})$ can be computed by gamma(*something*) using R.

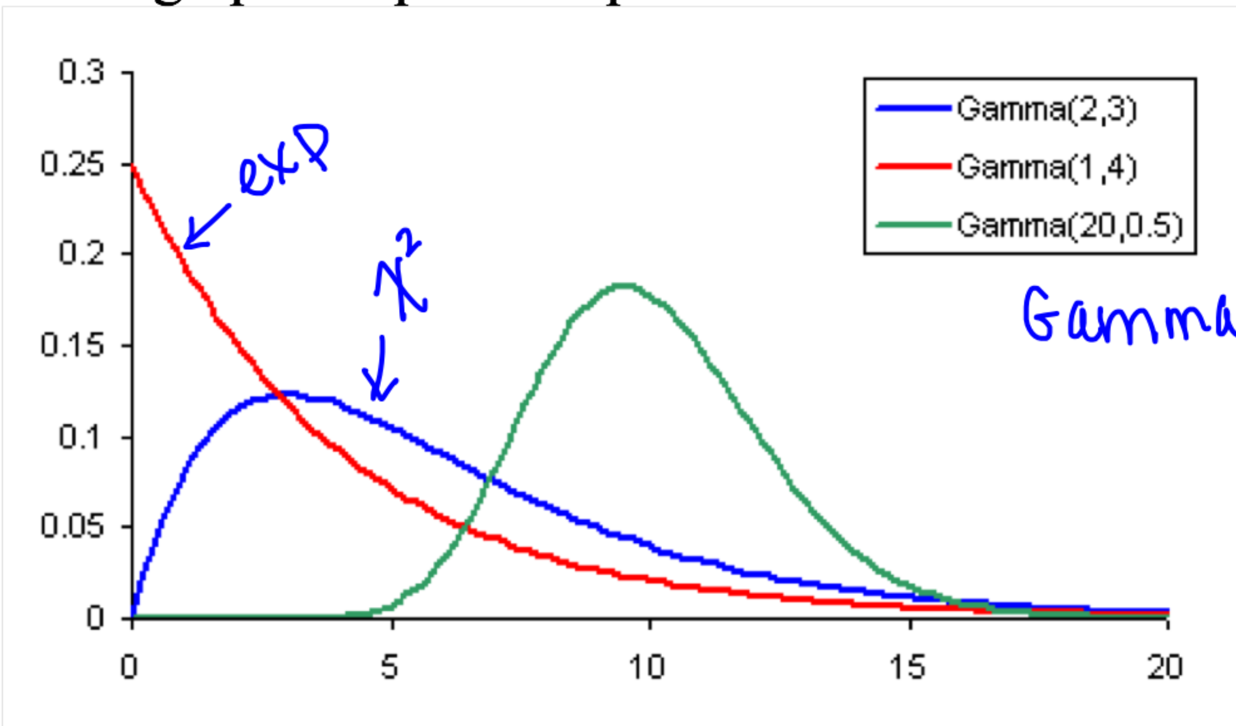
$$\boxed{E[X] = \mu = \alpha\beta \text{ and } V(X) = \sigma^2 = \alpha\beta^2}$$

Interpretation of the Gamma Distribution:

r.v.

If X has gamma distribution with parameters $\alpha > 0$ and $\beta > 0$, then X represents the amount of time it takes to obtain α successes, where $\beta = \frac{1}{\lambda}$, ($\lambda =$ expected number of occurrences is one time interval).

Some graphs of possible pdf's for the Gamma Distribution:



Gamma(α, β)

Special Cases of the Gamma Distribution:

1. If $\alpha = 1, \beta = \frac{1}{\lambda}$ then X is said to have *Exponential Distribution with parameter* λ.

2. If $\alpha = \frac{r}{2}, \beta = 2$ then X is said to have *Chi-square Distribution with parameter* r. In this case $f(y) = \frac{1}{\Gamma(r/2)2^{r/2}} y^{(r/2)-1} e^{-y/2}, \ \ 0 \le y < \infty$.

R commands
dgamma(x, shape, rate)
pgamma(x, shape, rate)
qgamma(p, shape, rate)
**shape = α
rate = $1/\beta$

$\mu = \alpha\beta \qquad \sigma^2 = \alpha\beta^2$

$12 = \alpha \cdot \beta \qquad 36 = \alpha\beta^2 = (\alpha\beta)\cdot\beta$

$36 = 12\beta$

$\alpha = 4 \qquad 3 = \beta$

Example: Suppose we are given a gamma distribution whose mean is 12 and standard deviation is 6. $\mu = 12$

$\sigma = 6 \qquad \sigma^2 = 36$

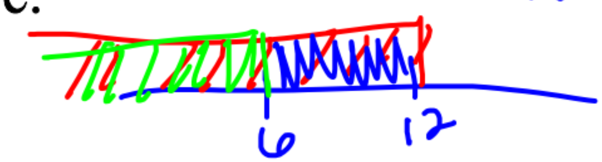
a. Find α and β

$\alpha = 4$, $\beta = 3$

b. $P(X \le 12) =$ pgamma(12, 4, 1/3) = .567 > .5

(mean: 12; median: .5)

c. $P(6 \le X \le 12) = P(X \le 12) - P(X \le 6) = .424$



d. Where is the median in relation to the mean?

$P(X \le \tilde{x}) = .5$

qgamma(.5, 4, 1/3)

$\tilde{x} = 11.016$

< 12

e. What is the 95th percentile for this distribution?

next page

$\mu = 12$

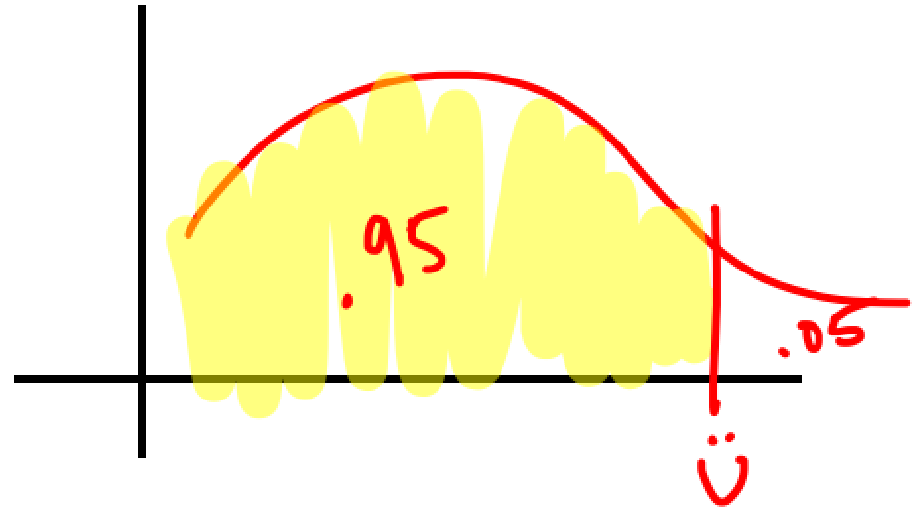
$P(X \le \bar{x} \text{ (mean)}) = .567$ ← calculated for this example

$P(X \le \tilde{x} \text{ (median)}) = .5$ ← for any distr.

if $P(X \le \text{mean}) > P(X \le \text{median})$

then mean > median

95th percentile



$P(X \leq c) = .95$

qgamma(.95, 4, 1/3) = 23.26

The Normal Distribution

If X has *normal distribution* with mean μ and variance σ^2, then X has pdf

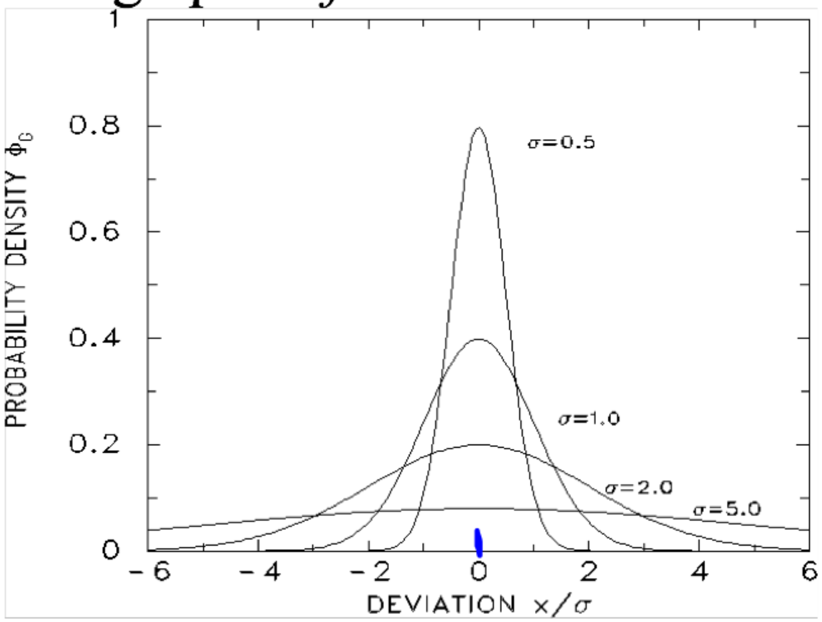
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < \infty.$$

and we write $X \sim N(\mu, \sigma^2)$.

Properties of the normal distribution:

1. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
2. $E[X] = \mu$.
3. $\operatorname{var}(X) = \sigma^2$.

The graph of f is well-known as the bell-shaped curve below.



mean = median

$\mu = 0$ $\sigma = 1$

Definition: The *standard normal random variable* is Z, where $Z \sim N(0,1)$.
and the cumulative distribution function for Z is Φ given by

$$F - \Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$
.

when $\mu = 0, \sigma = 1$

Example: If Z is the standard normal random variable, determine:

(a) $P(Z \le 2.15)$

```
> pnorm(2.15,0,1)
[1] 0.9842224
> pnorm(2.15)
[1] 0.9842224
```

in R studio

_norm(_ , μ , σ)

if standard normal

μ = 0 , σ = 1

R lets us leave out

(b) $P(2 < Z < 3)$ = pnorm(3) − pnorm(2)

= .0215

(c) $P(Z > 2)$ = 1 − pnorm(2) = .0228

(d) $P(-1 \le Z \le 1)$ = $P(|Z| \le 1)$ = pnorm(1) − pnorm(−1) = .6827

= $2 \cdot P(0 \le Z \le 1)$

(e) $P(-1.4 < Z < 2.01)$ = pnorm(2.01) − pnorm(−1.4)

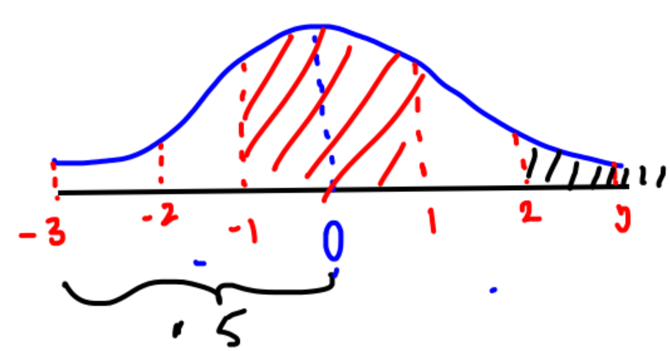
= .8970

(f) $P(Z > -1.57)$ = 1 − pnorm(−1.57)

= .9418

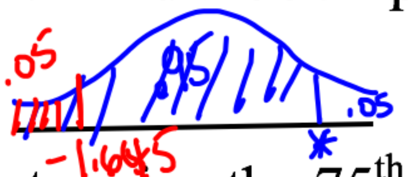
$P(Z < 1)$

$P(Z > 2$



In addition to calculations like those above, we can use the entries in the table to find the percentiles, z_p, such that $P(Z \leq z_p) = p$.

Ex: Find the 95th percentile, $z_{0.95}$. qnorm(.95) = 1.645



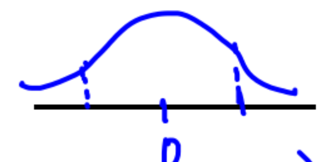
Determine the 75th, 50th and 99th percentiles.

qnorm(.75) = .6745

0

qnorm(.99) = 2.326

Can you use the 75th percentile to determine the 25th percentile?



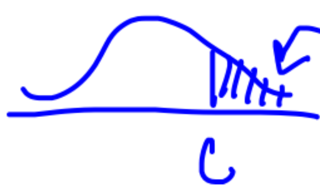
- qnorm(.75) = qnorm(.25)

Ex: Find a value of c so that $P(Z \leq c) = 0.7704$

qnorm(.7704)

Find a value of c so that $P(Z > c) = 0.006$.

qnorm(.994)



Find a value of c so that $P(-c \le Z \le c) = 0.966$

monday

Find a value of c so that $P(|Z| > c) = 0.05$

Popper 11

A forest products company claims that the amount of usable lumber in its harvested trees averages 172 cubic feet and has a standard deviation of 12.4 cubic feet. Assume that these amounts have approximately a normal distribution.

2. The median height of the trees is
   a. 165
   b. 178
   c. 150
   d. 172
   e. impossible to tell with the given information

3. If X is a *continuous* random variable, then $P(X \geq 2) =$

a. $1 - P(X \leq 2)$

b. $1 - P(X \leq 1)$

c. $P(X > 2)$

d. $1 - P(X < 3)$

e. none of these

4&5 = A