CUIN 7333

Notes: Sequences (part II)

More about recursive functions. Recall that we said a recursion formula is a formula that gives a_{n+1} in terms of one or more of the terms that precede a_{n+1}.

Examples:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = (n+1)a_n$$

$a_3 = a_{n+1}$ so $3 = n+1 \Rightarrow n = 2$
so $a_3 = (3) \cdot a_2$

a_1	a_2	a_3	a_4	a_5	a_6	a_7
1	$(2)(1) = 2$	$(3)(2) = 6$	$(4)(a_3) = 4 \cdot 6 = 24$	$(5)(a_4) = 5 \cdot 24 = 120$	$(6)(a_5) = 6 \cdot 120 = 720$	$(7)(a_6) = 7 \cdot 720 = 5040$

$a_2 = a_{n+1}$ so $2 = n+1$ and $n = 1$
in the formula

This example represents the factorial function

$$a_1 = a_2 = 1 \quad \text{and} \quad a_{n+2} = a_n + a_{n+1}$$

a_1	a_2	a_3	a_4	a_5	a_6
1	1	$a_3 = a_1 + a_2 = 1 + 1 = 2$	$a_4 = a_2 + a_3 = 1 + 2 = 3$	$a_5 = a_3 + a_4 = 2 + 3 = 5$	$a_6 = a_4 + a_5 = 3 + 5 = 8$

given

$a_3 = a_{n+2}$ so $n+2 = 3 \Rightarrow n = 1$

This second example is the Fibonacci sequence. It can actually be shown that $a_n = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n \sqrt{5}}$ and that if we found a sequence of ratios of consecutive terms $\left(\dfrac{a_{n+1}}{a_n}\right)$, the values would approach $\dfrac{2}{\sqrt{5}-1}$, the *golden ratio* ($\phi = 1.61803398875\ldots$). For more information on the golden ratio see here: http://www.jimloy.com/geometry/golden.htm

Finding a formula for a sequence:

The first several terms of a sequence $\{a_n\}$ are given. Assume that the pattern continues as indicated and find an explicit formula for a_n.

$0, \frac{3}{(2)}, \frac{8}{(3)}, \frac{15}{(4)}, \frac{24}{(5)}, \ldots$

$\leftarrow 3^2=9$ $\leftarrow 4^2=16$ $\leftarrow$ note $5^2=25$ all numerators are 1 less than the square of the denom.

$\underbrace{\frac{0}{1}, \frac{3}{2}, \frac{8}{3}, \frac{15}{4}, \frac{24}{5}}_{=n}, \ldots\ldots, \frac{99}{10} \leftarrow$ if $n=10$, $\ldots\ldots \frac{n^2-1}{n}$

what is numerator?

Sequences can be bounded or unbounded.

↑ have both a upper AND lower bound

The sequence $\{an\}$ is said to be

- *increasing* if $an < an+1$ for all n,
- *nondecreasing* if $an \le an+1$ for all n,
- *decreasing* if $an > an+1$ for all n,
- *nonincreasing* if $an \ge an+1$ for all n.

A sequence that satisfies any of these conditions is called *monotonic*.

Example: Determine the boundedness and monotonicity of the sequence with a_n as indicated.

$$\frac{(n+8)^2}{n^2}$$

Write out first few terms:

$\frac{9^2}{1^2}, \frac{10^2}{2^2}, \frac{11^2}{3^2}, \ldots\ldots, \underbrace{\frac{108^2}{100^2}, \ldots\ldots, \frac{10008^2}{10000^2}}, \ldots$

$9, 25, \frac{121}{9}, \ldots\ldots$

what happens to answer as n gets "really big"?

terms are decreasing (getting smaller)

Largest value $= 9$ (upper bounds)

no answer is smaller than 1 (lower bound)

} bounded

We are concerned with limits of sequences as n approaches infinity.
A sequence that has a limit is said to be *convergent*. A sequence that has no limit is said to be *divergent.*

Every convergent sequence is bounded and every unbounded sequence is divergent

THM - A bounded, nondecreasing sequence converges to its least upper bound; a bounded, nonincreasing sequence converges to its greatest lower bound.

Examples:

1. $\left\{\frac{1}{n^2}\right\} = 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

   decreasing
   upper bound = 1 bounded below by 0
   limit as $n \to \infty$ is 0

2. $\left\{\sqrt{4-\frac{1}{n}}\right\} = \sqrt{3}, \sqrt{3\tfrac{1}{2}}, \sqrt{3\tfrac{2}{3}}, \sqrt{3\tfrac{3}{4}}, \sqrt{3\tfrac{4}{5}}, \ldots, \sqrt{3\tfrac{99}{100}}$ ($\uparrow$ $n=100$)

   as $n \to \infty$ $a_n \to \sqrt{4} = 2$
   this sequence is increasing
   bounded below by $\sqrt{3}$ and above by 2

3. $\left\{\frac{n^2}{n+1}\right\} = \frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots, \frac{100}{11}, \ldots, \frac{10000}{101}$
   $(n=10)$ $(n=100)$

   increasing
   not bounded above
   bounded below by $1/2$

   the limit does <u>not exist</u>

Visualizing limits with Geogebra (free graphing software). http://www.geogebra.org/cms/