THE FUNCTION CONCEPT

INTRODUCTION.

Perhaps the single most important concept in mathematics is that of a *function.* However, the application and use of this concept goes far beyond "mathematics." At the heart of the function concept is the idea of a correspondence between two sets of objects. One of the definitions of "function" given in the Random House Dictionary of the English Language is: A factor related to or dependent on other factors: *price is a function of supply and demand*.

Correspondences between two sets of objects (functions) occur frequently in every day life.

Examples 1.1:

- To each item in supermarket there corresponds a price (item → price).
- To each student in a classroom there corresponds the chair that the student occupies (student → chair – i.e., the seating chart for the teacher).
- To each income earner in the U.S. in 2003 there corresponds the earner's federal income tax (income earner → federal income tax).
- To each registered student in a university there corresponds a grade-point average (student → GPA).

Here are some mathematical examples:

- To each real number x there corresponds its square x^2.
- Let S be the set of outcomes of a probability experiment. To each subset (event) E of S there corresponds the probability that E occurs, denoted by $P(E)$. ■

Each of these correspondences is an example of a function.

Definition of "Function": As suggested by the examples, a function consists of two sets of objects and a correspondence or rule that associates an object in one of the sets with an object in the other set,

(item → price)

(student → chair)

$$x \to x^2$$

$$E \to P(E)$$

and so on. However, it's not enough to state simply that a function is a correspondence that pairs the elements in one set with elements in another set. If you look carefully at the examples given above, you will note that each element in the first set is paired with exactly one element in the second set:

- Each supermarket item has exactly one price.
- Each student is sitting in exactly one chair.
- Each university student has exactly one GPA.
- Each real number x has exactly one square, x^2.

This is the key feature of the function concept.

DEFINITION 1: *A* function consists of two nonempty sets X and Y, and a rule f that associates each element x in X with one and only one element y in Y. This is symbolized by $f : X \to Y$ and read "the function f from X into Y." This phrase is often shortened to "the function f."

At this point it might help to give some examples of correspondences between two sets of objects that are not functions.

Examples 1.2:

- Let X be the set of rooms in a college dormitory, let Y be the set of students assigned to that dorm, and let the correspondence (rule) be

  room $\to$ student(s) assigned to the room.

  This is not a function because many rooms will have more than one student assigned. Also some rooms may have no students assigned (if the dorm is not full).

- Let X be the set of passengers on an airplane, let Y be the luggage in the baggage department, and let the correspondence be

  passenger $\to$ piece(s) of luggage.

  This is not a function because some passengers might not have any pieces of luggage while others may have several.

- Let X be the set of nonnegative real numbers, let Y be the set of all real numbers, and let the rule be

  $$x \rightarrow \text{the real numbers } y \text{ such that } y^2 = x$$

  This is not a function because to each x there are two y's. For example

  $$4 \rightarrow 2, -2; \quad 9 \rightarrow 3, -3; \quad 16 \rightarrow 4, -4; \quad \text{etc.}$$

- Let $X = Y$ be the set of all real numbers and let the rule be

  $$x \rightarrow y \text{ such that } y < x$$

  This is not a function because to each x there are infinitely many such y's – the set of all y's in $(-\infty, x)$. ■

If you think of a function as a "pairing" of elements in a set X with elements in another set Y, then the function concept can be defined in terms of ordered pairs.

DEFINITION 2: Let X and Y be sets. A *function* f *from* X *into* Y is a set S of ordered pairs (x, y), $x \in X$, $y \in Y$, with the property that (x, y_1) and (x, y_2) are in S if and only if $y_1 = y_2$.

To paraphrase this definition of "function," a set S of ordered pairs (x, y) is a function if and only if no two distinct ordered pairs in the set have the same first coordinate and *different* second coordinates.

Examples 1.3:

1. Let $S = \{(0,1), (1,2), (-1,2), (2,5), (-2,5), (3,10), (-3,10)\}$. Is S a function $f : X \rightarrow Y$? If so, what is X? What is the rule f?

2. Let $S = \{(0,0), (1,1), (2,-1), (1,2), (3,4), (3,6)\}$. Is S a function $f : X \rightarrow Y$? If so, what is X? What is the rule f?

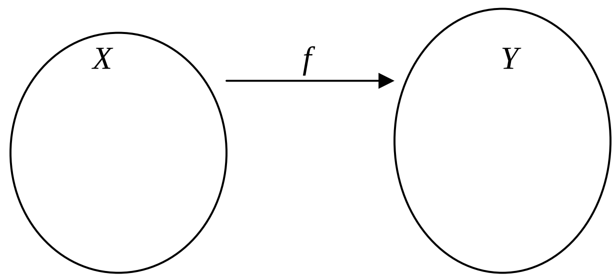
Solutions:

1. S is a function; no two ordered pairs have the same first coordinate and different second coordinates. $X = \{0, 1, -1, 2, -2, 3, -3\}$; a rule for this function is $x \rightarrow x^2 + 1$.

2. S is not a function; (1,1) and (1,2) are two ordered pairs in S with the same first coordinate and different second coordinates; (3,4) and (3,6) is another such pair. ■

TERMINOLOGY AND NOTATION

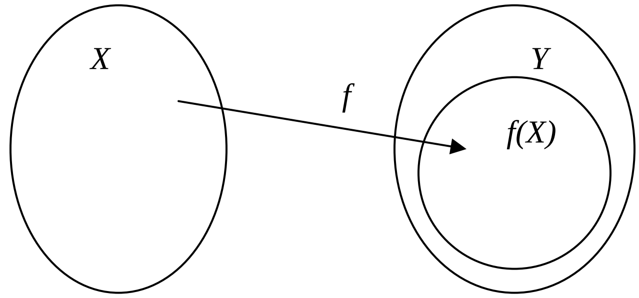
A figure such as



is often used to illustrate the function concept and the terminology that goes with it.

Domain and range: As suggested by the figure, a function "moves" from X to Y via some operation (rule) f: We choose an element x in X, perform the operation (apply the rule), and get an element y in Y. Indeed, by the definition, we get exactly one element y in Y. [If you take an item from the shelf in a super market, scan it (the rule), you get the (unique) price.] The first set X is called the *domain* of the function. The set of y's in Y which correspond to the elements x in X is called the *range* of the function. The range of f is denoted by $f(X)$.

The range of a function $f: X \to Y$, is a subset of Y. In general, it is a proper subset; typically there will be y's in Y, which do not correspond to *any* x in X. For example, in our classroom example above, there may be extra chairs in the room; chairs that are not occupied by any student.



Here are some other examples.

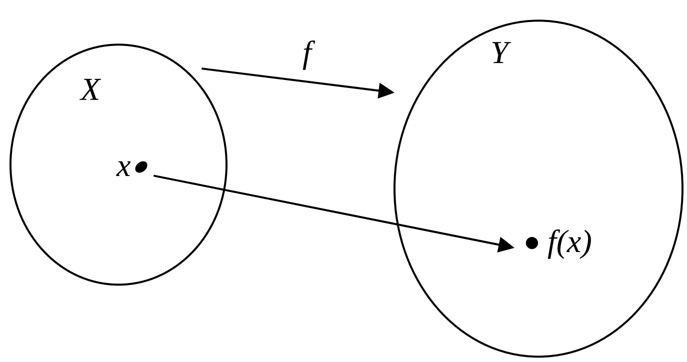
Examples 1.4:

1. Let $X = Y$ = the set of real numbers, and let f be the squaring function, $f : x \to x^2$. The range of f is the set of nonnegative real numbers; no negative number is in the range of this function.

2. Consider a university with 25,000 students. Let X be the students enrolled in the university, let Y be the set of 4-decimal place numbers 0.0000 to 4.0000, and let f be the rule: student $\to$ GPA. Since there are 50,000 4-decimal place numbers and only 25,000 students there will be at least 25,000 numbers (probably more) which are not in the range of this function. ■

Independent and dependent variables: Symbols (such as x and y) used to represent elements in the domain and range of a function f are called *variables*. More specifically, a variable, such as x, used to represent an element in the domain is called an *independent variable*, and a variable, such as y, used to represent the corresponding element in the range is called a *dependent variable*. This terminology is derived from the fact that we can choose an element x in the domain arbitrarily (independently). Then, applying the rule f to x, we get an element y in the range of f; the variable y "depends" on the variable x.

Function notation: Suppose $f : X \to Y$ is a given function. If we choose an element x in X, apply the rule f, we get an element in Y. The symbol $f(x)$ is used to represent the element in Y that corresponds to the element $x \in X$. This symbol is read in a variety of ways – "f of x", "the value of f at x", and "the image of x under f" – are some of the more common readings. In this notation, our figure becomes



Examples 1.5:

1. Let $X = Y =$ the set of real numbers, and let f be the rule $f : x \to x^2$ (f is the squaring function). Some of the values of f are:

$$f(2) = 2^2 = 4, \quad f(-2) = (-2)^2 = 4, \quad f(3) = 3^2 = 9, \quad f(-3) = (-3)^2 = 9 \quad \text{and so on.}$$

In general, to find $f(a)$ we simply replace x by a and calculate a^2. That is, $f(a) = a^2$. For this reason, the squaring function is usually denoted by the equation $f(x) = x^2$.

2. Let $X = Y =$ the set of real numbers, and let g be the function, $g : x \to x^3 - x + 2$. This function is denoted by the equation $g(x) = x^3 - x + 2$. Some of the values of g are:

$$g(-1) = (-1)^3 - (-1) + 2 = 2, \quad g(3) = (3)^3 - (3) + 2 = 27 - 3 + 2 = 26,$$

$$g(-2) = (-2)^3 - (-2) + 2 = -8 + 2 + 2 = -4, \; g(0) = 0^3 - 0 + 2 = 2, \; g(1) = 1^3 - 1 + 2 = 2.$$

3. Let $X = Y =$ the set of real numbers, and let h be the function given by the equation: $h(x) = x^2 - 2x + 3$. For real numbers a and b

$$h(a) = a^2 - 2a + 3$$

and

$$h(a+b) = (a+b)^2 - 2(a+b) + 3 = a^2 + 2ab + b^2 - 2a - 2b + 3. \quad \blacksquare$$

One-to-one functions: If $f : X \to Y$, then to each $x \in X$ there corresponds a unique element $y = f(x) \in Y$. Now suppose we pick an element $z \in Y$. Is there an element $x \in X$ such that $f(x) = z$? "Not necessarily", as we saw in our examples above; the range of f, is, in general, a proper subset of Y; the element z that we selected may not be in the range of f. However, if z is in the range of f, then "yes" there is an element $x \in X$ such that $f(x) = z$. Could there be more than one x such that $f(x) = z$? In general, "yes". Look at the functions $f(x) = x^2$ and $g(x) = x^3 - x$.

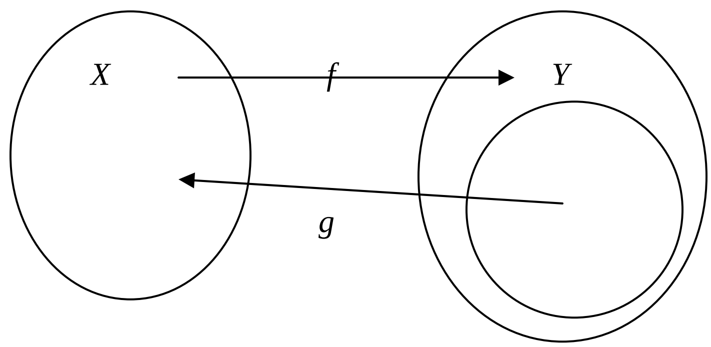
$$f(x) = x^2: \quad f(-2) = f(2) = 4; \qquad g(x) = x^3 - x: \quad g(-1) = g(0) = g(1) = 0.$$

Two distinct points map to 4 under f; three distinct points map to 0 under g.

While the definition of "function" requires that to each x in the domain there corresponds one and only one y in the range, the definition *does not* require that each y in the range be the image of one and only one x in the domain.

A function $f: X \to Y$ that has the special property that each y in the range is the image of one and only one x in the domain is called a *one-to-one* function. A function f is one-to-one if and only if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. Another way to say this is: f is one-to-one if and only if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Suppose $f: X \to Y$ is one-to-one. Since to each $z \in f(X)$ (the range of f) there corresponds one and only one $x \in X$ such that $f(x) = z$, we can define a function $g: f(X) \to X$ by $g(z) = x$.



This function g: $f(X) \to X$ is called the *inverse function of* f and is denoted by f^{-1}. Some concrete examples of functions and inverse functions will be given in the sections that follow.

Examples 1.7:

1. Let f be the function: supermarket item $\to$ price. Is f one-to-one?

2. Let f be the function: student $\to$ GPA. Is f one-to-one?

3. Let f be the function:

   car $\to$ license plate (letter,letter,letter – number,number,number)
   Is f one-to-one?

4. Let $X = Y =$ the set of real numbers and let f be the function $f(x) = 2x + 3$. Show that f is one-to-one?

Solutions:

1. f is not one-to-one; different items can have the same price.

2. f is not one-to-one; different students can have the same GPA

3. f is one-to-one; different cars have different license plates.

4. Assume f is not one-to-one. Then there must exist real numbers x_1 and x_2, $x_1 \neq x_2$, such that

$$2x_1 + 3 = 2x_2 + 3.$$

It follows from this equation that $x_1 = x_2$. This contradicts our assumption that f is not one-to-one. Therefore, f is one-to-one. ■

Exercises:

1. If $f(x) = 2x^2 - 3x + 4$, find $f(1)$, $f(-1)$, $f(0)$, and $f(2)$.

2. If $f(x) = x^3 + 5x^2 - 1$, find $f(2)$, $f(-2)$, $f(0)$, and $f(-1)$.

3. If $f(x) = \sqrt{x-1} + 2x$, find $f(1)$, $f(3)$, $f(5)$, and $f(10)$.

4. If $f(x) = 2x^2 - x + 3$, find
   a. $f(c)$ b. $f(-c)$ c. $-f(c)$

   d. $f(c + h)$ e. $f(c) + f(h)$

5. If $g(x) = 3x - 8$, find
   a. $g(1/a)$ b. $1/g(a)$ c. $g(a^2)$

   d. $[g(a)]^2$

In each of the following, determine whether the number 4 is in the range of f. If it is, find all the numbers a in the domain of f such that $f(a) = 4$.

6. $f(x) = 6x - 5$.

7. $f(x) = 3 - 2x$.

8. $f(x) = \sqrt{x-3}$.

9. $f(x) = 1/x.$

10. $f(x) = x^2 + 5.$

11. Which of the following sets of order pairs is a function?
    a. {(4,2), (4,3), (3,4), (2,5), (1,4)}
    b. {(1,7), (2,8), (3,9), (4,10), (5,12)}
    c. {(1,3), (2,3), (3,3), (4,3), (5,3)}

12. Which of the following sets of order pairs is a function?
    a. {(1,2), (2,2), (3,4), (4,4), (4,5)}
    b. {(1,3), (5,1), (5,2), (3,1), (4,0)}
    c. {(95,15), (72, 37), (10,30), (15, 45)}

In each of the following, determine whether f is one-to-one.

13. $f(x) = 3x + 2$.

14. $f(x) = 5 - 3x.$

15. $f(x) = 1/(2x + 1).$

16. $f(x) = x^2 - 2x + 1.$

17. Find a function that expresses the radius r of a circle as a function of its circumference C. If the circumference of a circle is increased by 6 cm, determine how much the radius increases.

18. Find a function that expresses the perimeter P of a square as a function of its area A.

19. Find a formula that expresses the volume V of a cube as a function of its surface area S. Find the volume if the surface area is 36 sq. in.

20. Your car gets 30 miles per gallon. You fill up the tank with 15 gallons. Find a function that expresses the number of gallons g left in the tank after you drive m miles.

FUNCTIONS IN MATHEMATICS

As indicated in the Introduction, the function concept is, perhaps, the most important concept in mathematics. Almost all mathematical theory involves functions in one form or another.

In the modules on Geometry and Measurement we consider functions f whose domains are geometric objects and whose range is a set of real numbers. For example:

- The length function: x = "the line segment from point A to point B", $f(x)$ = the length of the line segment.
- The area function: x = "the rectangle with length l and width w", $f(x)$ = the area of x.

And so on.

In the module on probability we consider functions P whose domain is the set of subsets of a set S (the sample space of a probability experiment) and whose range is a set of real numbers in the interval $[0,1]$:

- If $E \subseteq S$, then $P(E) =$ the probability that event E occurs.

Real-valued functions of a real variable: In this module we will be studying functions $f : X \to Y$ where Y is the set of real numbers and X is some subset of the set of real numbers. Such functions are called *real-valued functions of a real variable.* Hereafter when we use the term "function" we will mean a function of this specific type. Functions will usually be denoted by letters such as *f, g, h, F,* etc.

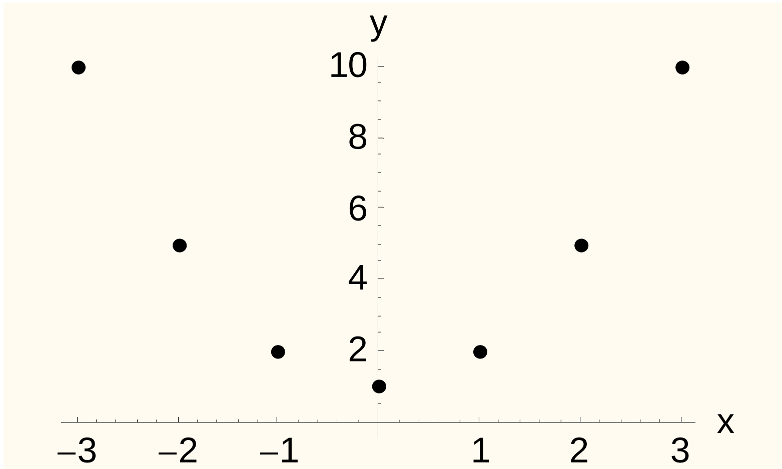
Graph of a function: Let f be a function with domain X. The *graph of* f is a geometric representation (a "picture") of f; it is a set of points plotted in the coordinate plane. The graph of f is defined as follows:

$$\text{graph of } f = \{(x, f(x)) \mid x \in X\}.$$

Another way to describe the graph of a function is: The graph of a function f is the graph of the equation $y = f(x)$, x in the domain of f.

Examples 2.1:

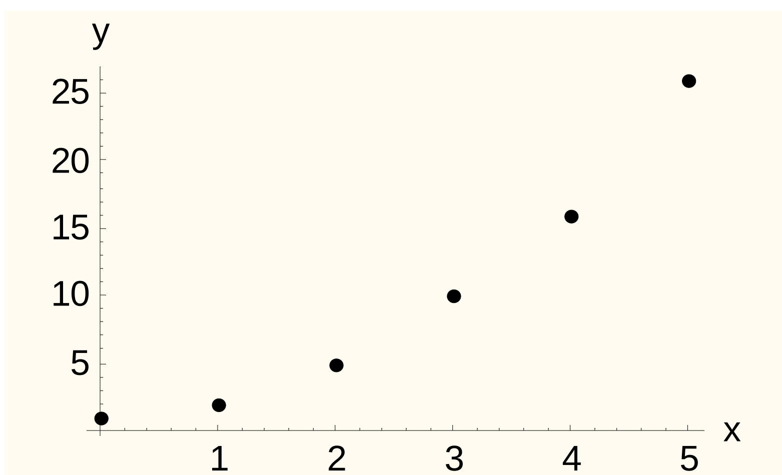
1. Let f be the function $\{(0,1), (1,2), (-1,2), (2,5), (-2,5), (3,10), (-3,10)\}$ (see Example 1.3.1). The graph of f is



Recall from Example 1.3.1 that a rule for f is $f(x) = x^2 + 1, \quad x \in \{-3, -2, \ldots, 3\}$.

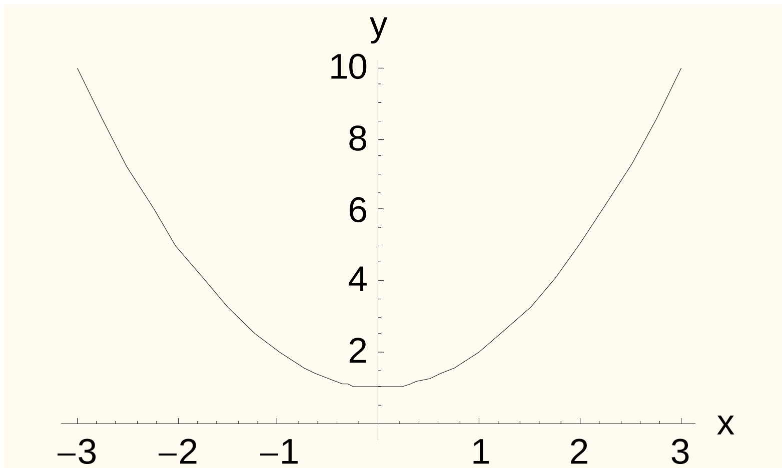
2. Let g be the function with domain $J = \{1, 2, 3, 4, \ldots\}$ (the set of positive integers), and $g(n) = n^2 + 1$. The graph of g is

N	*G(n)*
1	2
2	5
3	10
4	17
5	26



Note: In (1) we plotted all of the pairs in *f*. We are not able to plot *all* the pairs here, there are infinitely many of them, but we plotted enough to get the idea what the function looks like.

3. Let X = the set of real numbers and let h be the function given by $h(x) = x^2 + 1$. the graph of h is



Here we simply used the points that we plotted in (1), connected them with a smooth curve, following the pattern indicated by the points. $\blacksquare$

Note the graphs in Examples 2.1. The graphs in Examples 2.1.1 and 2.1.2 are discrete points in the plane because their domains are discrete points on the real line. In Example 2.1.3 the graph is a (continuous) curve because the domain is an interval (the whole real line in this case).

Other than the examples already given, we will not consider functions with domain X a finite set of real numbers. We want to focus our attention on functions whose domains are the set of positive integers and on functions whose domains are intervals on the real line.