Quadratic Functions

Quadratic functions are one of the simplest types of nonlinear functions. They can be written in the form

$$f(x) = ax^2 + bx + c$$

where $a \neq 0$.

Examples:

(1) The function $f(x) = x^2$ is a quadratic function corresponding to the values $a = 1$, $b = 0$ and $c = 0$.

(2) The function $g(x) = -\frac{1}{2}x^2 + 3x$ is a quadratic function corresponding to the values $a = -1/2$, $b = 3$ and $c = 0$.

The expression $ax^2 + bx + c$ which arises in the definition of a quadratic function is called an **quadratic expression**.

A. Factoring Quadratic Expressions

Sometimes **quadratic expressions** $ax^2 + bx + c$ arise from the product of two linear factors. For example, consider the product $(x-2)(x+3)$. We can use the distributive law to expand this product as

$$\begin{aligned}(x-2)(x+3) &= x(x+3) - 2(x+3) \\ &= x^2 + 3x - 2x - 6 \\ &= x^2 + x - 6\end{aligned}$$

Consequently, we can see that the quadratic expression $x^2 + x - 6$ is the product of the linear factors $(x-2)$ and $(x+3)$.

Examples:

1. We can expand the product $(x+7)(x-4)$ using the distributive law.

$$\begin{aligned}(x+7)(x-4) &= x(x-4) + 7(x-4) \\ &= x^2 - 4x + 7x - 28 \\ &= x^2 + 3x - 28\end{aligned}$$

As a result, the quadratic expression $x^2+3x-28$ is the product of the linear factors $(x+7)$ and $(x-4)$.

2. We can expand the product $(2x+1)(3x-4)$ using the distributive law.

$$\begin{aligned}(2x+1)(3x-4) &= 2x(3x-4)+1(3x-4)\\ &= 6x^2-8x+3x-4\\ &= 6x^2-5x-4\end{aligned}$$

As a result, the quadratic expression $6x^2-5x-4$ is the product of the linear factors $(2x+1)$ and $(3x-4)$.

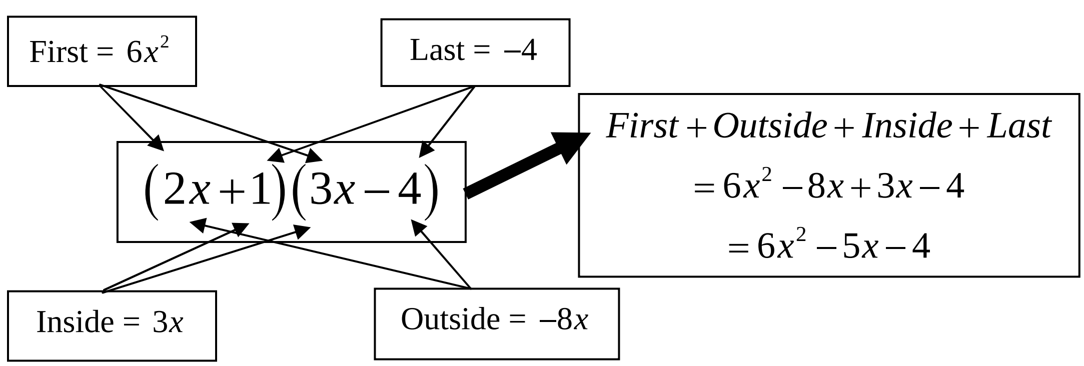
Remark: There are a number of methods for multiplying linear factors. All of these methods ultimately use the distributive property. We illustrate two of these below.

1. **The Table Method**: Consider the product $(2x+1)(3x-4)$ from the previous example. Start by creating the table shown below.

Step 1: Create the table on the right.	$\times$	$3x$	-4
	$2x$	$6x^2$	$-8x$
	1	$3x$	-4
Step 2: Sum the shaded entries.		$6x^2+3x-8x-4$	
Step 3: Simplify the sum.		$6x^2-5x-4$	

The entry on the bottom right of the table is the product $(2x+1)(3x-4)$.

2. **The FOIL Method:** The letters in the word "FOIL" stand for First, Outside, Inside, Last. We illustrate this in the figure below with the product $(2x+1)(3x-4)$.



In most cases, we already know the quadratic expression and we need to determine whether it can be written as a the product of linear factors. The process of finding the linear factors is called **factoring**. We start with quadratic expressions of the form x^2+bx+c; i.e. ones with a coefficient of 1 in front of the x^2 term. In order for x^2+bx+c to be written as the product $(x+p)(x+q)$ we need

$$
\begin{aligned}
x^2+bx+c &= (x+p)(x+q) \\
&= x(x+q)+p(x+q) \\
&= x^2+qx+px+pq \\
&= x^2+(p+q)x+pq
\end{aligned}
$$

At first glance, this might seem impossible to use. However, notice that the first and last expressions tell us that p and q need to be chosen so that

$$p+q=b \text{ and } pq=c$$

That is, the sum of p and q is b, and the product of p and q is c.

Example: Use the method above to factor x^2+5x+4.

According to the discussion, we can write

$$x^2+5x+4=(x+p)(x+q)$$

if and only if we can find numbers p and q so that

$$p+q=5 \text{ and } pq=4$$

Notice that p and q have the same algebraic sign since their product is positive. Furthermore, the algebraic sign must be positive since they add to 5. The table below is helpful.

***p* and *q* so that *pq* = 4**	***p* + *q***
2 and 2	4
1 and 4	5

The last row of the table tells us we can use $p=1$ and $q=4$. Consequently, x^2+5x+4 can be factored as

$$x^2+5x+4=(x+1)(x+4)$$

Example: Use the method above to factor $x^2-3x-54$.

According to the earlier discussion, we can write $x^2-3x-54=(x+p)(x+q)$ if and only if we can find numbers p and q so that

$$p + q = -3 \text{ and } pq = -54$$

Notice that p and q have different algebraic signs since their product is negative. The table below is helpful.

***p* and *q* so that *pq* = -54**	***p* + *q***
1 and -54	-53
-1 and 54	53
2 and -27	-25
-2 and 27	25
3 and -18	-15
-3 and 18	15
6 and -9	-3

We could have included more rows in the table since -54 has other factors (e.g., -6 and 9). However, The last row of the table tells us we can use $p = 6$ and $q = -9$. Consequently, $x^2 - 3x - 54$ can be factored as

$$x^2 - 3x - 54 = (x+6)(x-9)$$

<u>Example</u>: Sometimes it is not possible to factor a quadratic polynomial as the product of two linear factors with integer coefficients. For example, consider the problem of determining whether it is possible to factor $x^2 + 5x + 12$ as a product of linear factors with integer coefficients.

According to the discussion above, we can write $x^2 + 5x + 12 = (x+p)(x+q)$ if and only if we can find numbers p and q so that

$$p + q = 5 \text{ and } pq = 12$$

Notice that p and q have the same algebraic sign since their product is positive. Furthermore, the algebraic sign must be positive since they add to 5. The table below is helpful.

***p* and *q* so that *pq* = 12**	***p* + *q***

1 and 12	13
2 and 6	8
3 and 4	7

We can see from the table that it is not possible to find integers *p* and *q* satisfying the conditions. Consequently, it is not possible to factor $x^2+5x+12$ as a product of linear factors with integer coefficients.

It is a little more complicated to factor ax^2+bx+c when $a \neq 1$. Suppose we want to write ax^2+bx+c as $(rx+p)(sx+q)$. The calculation below will help us organize our work.

$$\begin{aligned} ax^2+bx+c &= (rx+p)(sx+q) \\ &= rx(sx+q)+p(sx+q) \\ &= rsx^2+rqx+psx+pq \\ &= rsx^2+(ps+rq)x+pq \end{aligned}$$

From above, we can find values for *r*, *s*, *p* and *q* if and only if

$$rs=a,\ ps+rq=b,\ \text{and } pq=c$$

<u>Example</u>: Determine whether it is possible to factor $2x^2+5x-12$ as a product of the form $(rx+p)(sx+q)$ where *r*, *s*, *p* and *q* are integers.

Notice that *r* and *s* have the same algebraic sign since their product is positive, and *p* and *q* have the opposite algebraic sign since their product is negative. The following table is helpful.

***r* and *s* with *rs* = 2**		***p* and *q* with *pq* = -12**		**We need** $ps+rq=5$
r	***s***	***p***	***q***	$ps+rq$
1	2	1	-12	-10
2	1	1	-12	-23
1	2	-1	12	10
2	1	-1	12	23
1	2	-3	4	-2
2	1	-3	4	5

There are other possible combinations, but the last row of the table already tells us

$$2x^2 + 5x - 12 = (2x - 3)(x + 4)$$

We can use the distributive law to verify this result.

$$\begin{aligned}(2x-3)(x+4) &= 2x(x+4) - 3(x+4)\\ &= 2x^2 + 8x - 3x - 12\\ &= 2x^2 + 5x - 12\end{aligned}$$

One special factoring is a **difference of squares**. We can illustrate this with the following calculation.

$$\begin{aligned}(x-a)(x+a) &= x(x+a) - a(x+a)\\ &= x^2 + ax - ax - a^2\\ &= x^2 - a^2\end{aligned}$$

Notice that the last line above is a difference of the squares x^2 and a^2, and the quadratic expression $x^2 - a^2$ factors as $(x-a)(x+a)$.

<u>Example</u>: The quadratic expression $x^2 - 4$ is a difference of squares since $2^2 = 4$. Consequently, from the discussion above

$$x^2 - 4 = (x-2)(x+2)$$

Similarly, $x^2 - 9 = (x-3)(x+3)$ and $x^2 - 1 = (x-1)(x+1)$. Notice that it is not necessary for a^2 to be the square of an integer. For example, $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$.

Factoring can sometimes be used to solve **quadratic equations**. Suppose we want to solve $ax^2 + bx + c = 0$ and we know that $ax^2 + bx + c$ can be factored as $(rx + p)(sx + q)$. Then $ax^2 + bx + c = 0$ is equivalent to $(rx + p)(sx + q) = 0$. Now, notice that a product can only be zero if at least one of the factors is zero. Consequently,

$$ax^2 + bx + c = 0 \text{ if and only if either } rx + p = 0 \text{ or } sx + q = 0$$

<u>Example</u>: We can use factoring to solve the equation $x^2 + 5x + 4 = 0$. We saw earlier that

$$x^2 + 5x + 4 = (x+4)(x+1)$$

Consequently,

$$x^2+5x+4=0 \text{ if and only if either } x+4=0 \text{ or } x+1=0$$

The solutions of these last two linear equations are $x=-4$ and $x=-1$. Therefore, the solutions to $x^2+5x+4=0$ are $x=-4$ and $x=-1$.

Exercises:

1. Use the distributive property to expand the product $(-2x+1)(3x-5)$.
2. Use the distributive property to expand the product $(x-7)(5x+6)$.
3. Use the table method to expand the product $(-2x+1)(3x-5)$.
4. Use the table method to expand the product $(x-7)(5x+6)$.
5. Use the FOIL method to expand the product $(-2x+1)(3x-5)$.
6. Use the FOIL method to expand the product $(x-7)(5x+6)$.
7. Expand the product $(-2x+1/2)(5x-2)$.
8. Expand the product $(10x+1)(10x-1)$.
9. Expand the product $(ax+b)(ax-b)$.
10. Factor (if possible) the quadratic expression x^2+3x-4.
11. Factor (if possible) the quadratic expression $3x^2+4x+1$.
12. Factor (if possible) the quadratic expression x^2-16.
13. Factor (if possible) the quadratic expression x^2-100.
14. Factor (if possible) the quadratic expression x^2+100.
15. Factor (if possible) the quadratic expression x^2+x+6.
16. Factor (if possible) the quadratic expression $a^2x^2-b^2$.
17. Use factoring to find all solutions of the equation $x^2+3x-4=0$.
18. Use factoring to find all solutions of the equation $3x^2+4x+1=0$.
19. Use factoring to find all solutions of the equation $x^2-16=0$.
20. Use factoring to find all solutions of the equation $x^2-100=0$.
21. Use factoring to find all solutions of the equation $a^2x^2-b^2=0$.

B. Graphs of Quadratic Functions

We can understand the behavior of every quadratic function by studying the functions

$$f(x)=x^2 \text{ and } g(x)=-x^2$$

We will see that every other quadratic function is a scaling and shift of one of these two functions.

Both $f(x)=x^2$ and $g(x)=-x^2$ have a nice symmetry property. Namely, f and g treat positive numbers the same as they treat their negative counterparts. We can see this with a few computations.

$f(1)=1^2=1=(-1)^2=f(-1)$ $f(2)=2^2=4=(-2)^2=f(-2)$ $f(3)=3^2=9=(-3)^2=f(-3)$	$g(1)=-1^2=-1=-(-1)^2=g(-1)$ $g(2)=-2^2=-4=-(-2)^2=g(-2)$ $g(3)=-3^2=-9=-(-3)^2=g(-3)$

In general,

$$f(x)=x^2=(-x)^2=f(-x)$$

and

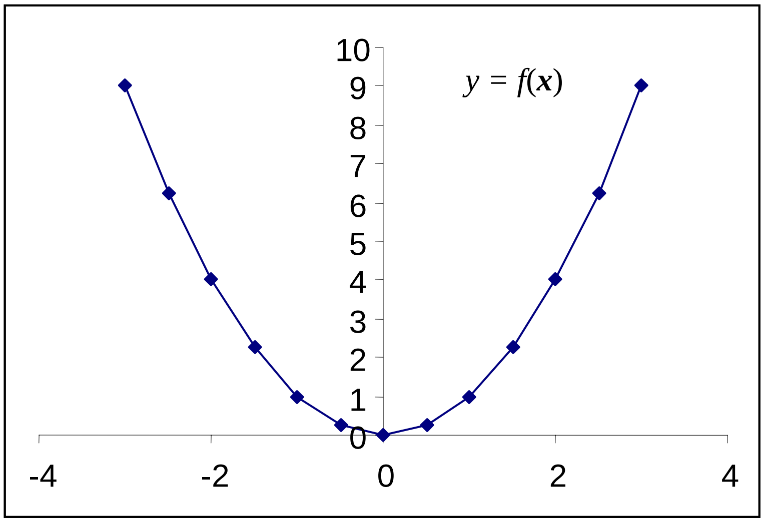
$$g(x)=-x^2=-(-x)^2=g(-x)$$

The information above tells us the following points are on the graphs of *f* and *g*.

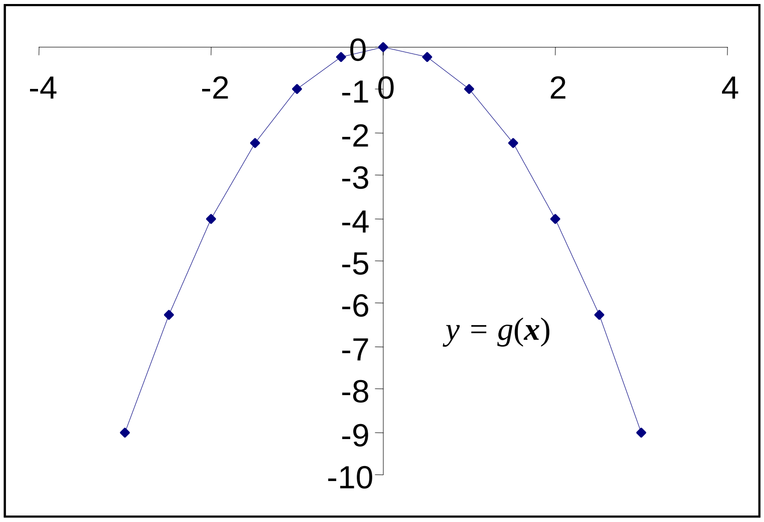
Points on the graph of *f*	**Points on the graph of *g***
$(1,1)$ and $(-1,1)$ $(2,4)$ and $(-2,4)$ $(3,9)$ and $(-3,9)$ in general.... (x,x^2) and $(-x,x^2)$	$(1,-1)$ and $(-1,-1)$ $(2,-4)$ and $(-2,-4)$ $(3,-9)$ and $(-3,-9)$ in general.... $(x,-x^2)$ and $(-x,-x^2)$

As a result, **the graphs of both *f* and *g* are symmetric about the *y*-axis**. That is, the graphs for positive values of *x* are the mirror image of the graphs for negative values of *x*. The tables and graphs show this below for *f* and *g*.

X	$f(x)$
-3	9
-2.5	6.25
-2	4
-1.5	2.25
-1	1
-0.5	0.25
0	0
0.5	0.25
1	1
1.5	2.25
2	4
2.5	6.25
3	9



X	$g(x)$
-3	-9
-2.5	-6.25
-2	-4
-1.5	-2.25
-1	-1
-0.5	-0.25
0	0
0.5	-0.25
1	-1
1.5	-2.25
2	-4
2.5	-6.25
3	-9



Now we are ready to show how every quadratic function can be obtained from x^2 or $-x^2$ by using shifts and scaling. We do this first for a specific example.

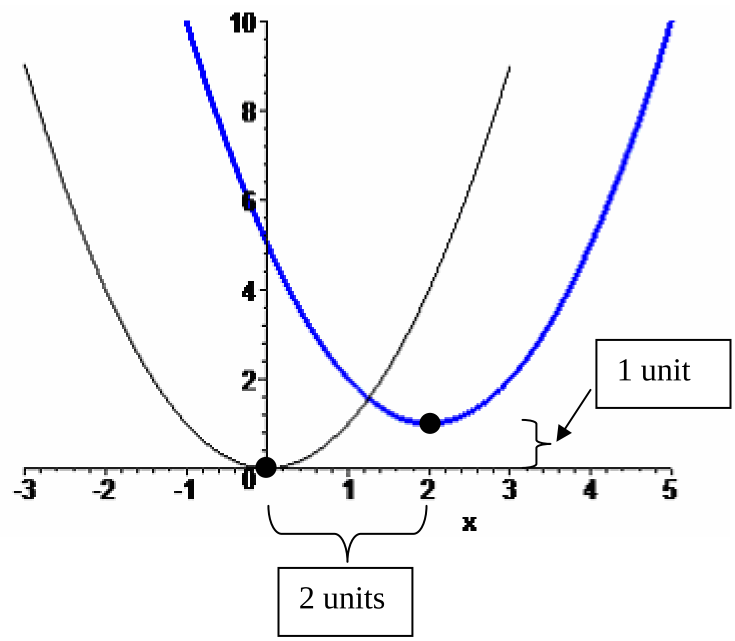
Example: Let $f(x) = x^2 - 4x + 5$. We can rewrite the formula for f as follows.

$$\begin{aligned} f(x) &= x^2 - 4x + 5 \\ &= \left(x^2 - 4x + \Delta\right) + 5 - \Delta \end{aligned}$$

Notice that $(x-2)^2 = x^2 - 4x + 4$. So, replacing Δ above with 4 gives

$$
\begin{aligned}
f(x) &= x^2 - 4x + 5 \\
&= \left(x^2 - 4x + 4\right) + 5 - 4 \\
&= \left(x - 2\right)^2 + 1
\end{aligned}
$$

Notice that the final expression is a horizontal and vertical shift of $f\left(x\right) = x^2$. As a result, the graph of $f(x) = x^2 - 4x + 5$ can be obtained by shifting the graph of $f\left(x\right) = x^2$ by 2 units to the in the *x* direction and by 1 unit in the *y* direction. The graph is shown below along with the graph of $f\left(x\right) = x^2$.



Now we repeat the process above for the general quadratic function given by

$$f(x) = ax^2 + bx + c$$

where $a \neq 0$.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x + \Delta\right) + c - a\Delta \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\frac{b^2}{4a^2} \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\ &= a\left(x - \frac{-b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

As a result, if $a > 0$ the graph of *f* can be obtained from the graph of x^2 using the following steps:

- scale x^2 by the value *a* to obtain ax^2
- shift ax^2 by $\frac{-b}{2a}$ units in the ***x*** direction and $c - \frac{b^2}{4a}$ units in the ***y*** direction.

Notice that in this process, the point $(0,0)$ on the graph of x^2 is translated to the point $\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right)$ to obtain the graph of $f(x) = ax^2 + bx + c$.

Similarly, if $a < 0$ the graph of *f* can be obtained from the graph of x^2 using the following steps:

- scale $-x^2$ by the value -*a* to obtain ax^2 (remember, $a < 0$)
- shift ax^2 by $\frac{-b}{2a}$ units in the ***x*** direction and $c - \frac{b^2}{4a}$ units in the ***y*** direction.

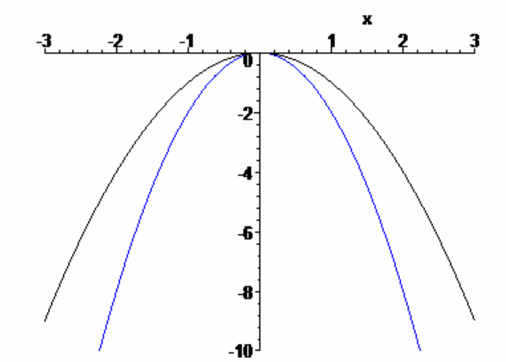
Notice that in this process, the point $(0,0)$ on the graph of x^2 is translated to the point $\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right)$ to obtain the graph of $f(x) = ax^2 + bx + c$.

From above, all of the graphs of quadratic functions have common shape (which is either oriented upwards or downwards).

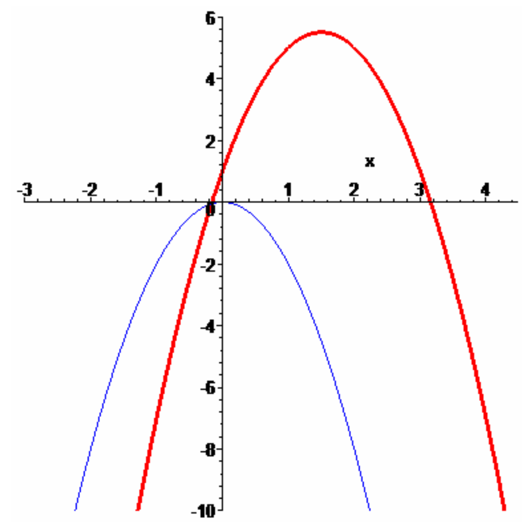
<u>Example</u>: Let $f(x) = -2x^2 + 6x + 1$. We can rewrite the formula for *f* as follows.

$$\begin{aligned} f(x) &= -2x^2 + 6x + 1 \\ &= -2\left(x^2 - 3x + \Delta\right) + 1 - (-2)\Delta \\ &= -2\left(x^2 - 3x + \left(\frac{3}{2}\right)^2\right) + 1 - (-2)\left(\frac{3}{2}\right)^2 \\ &= -2\left(x - \frac{3}{2}\right)^2 + 1 + \frac{9}{2} \\ &= -2\left(x - \frac{3}{2}\right)^2 + \frac{11}{2} \end{aligned}$$

The final expression is a scaling, horizontal shift, and vertical shift of $f(x) = -x^2$. More precisely, the graph of $f(x) = -2x^2 + 6x + 1$ can be obtained by scaling the graph of $f(x) = -x^2$ by a factor of 2, and then shifting the resulting graph by 3/2 units in the *x* direction and by 11/2 units in the *y* direction. The graph is shown below as a two step process. First we show the graph of $f(x) = -x^2$ along with its scaling by a factor of 2. Then we show the shifts of the resulting graph.



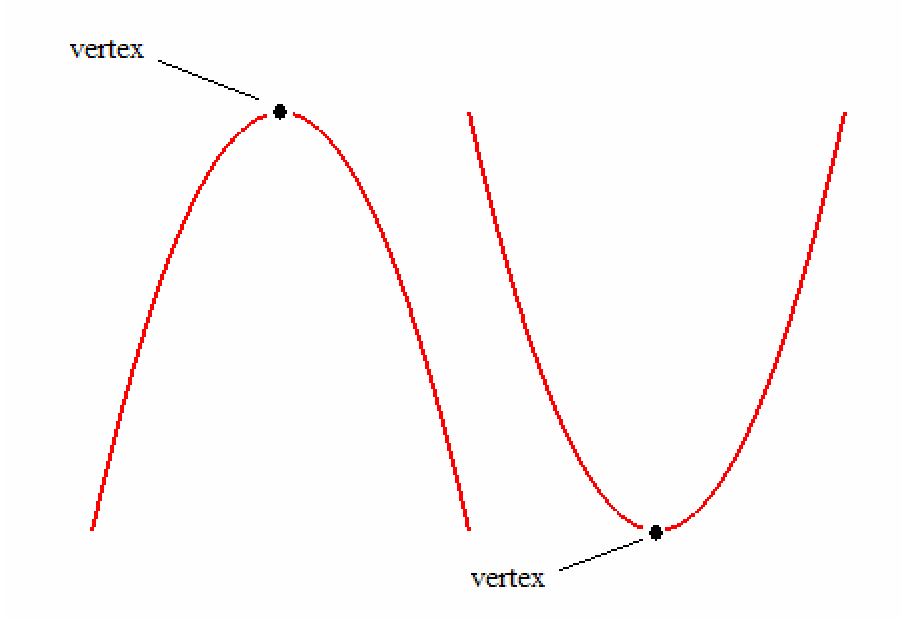
Scaling $f(x) = -x^2$ by a factor of 2.



Shifting 3/2 units in the *x* direction and 11/2 units in the *y* direction.

The common shape of graphs of quadratic functions is called a **parabola**.

Since all parabolas are similar in shape to the graphs of $f(x) = x^2$ and $g(x) = -x^2$, they each have one of the shapes sown below.



Parabolas which arise as the graph of $f(x) = ax^2 + bx + c$ with $a < 0$ have the shape on the left above, whereas those that arise with $a > 0$ have the shape on the right above. The point labeled as the **vertex** in each of the figures above are given by (0,0) in the graphs of $f(x) = x^2$ and $g(x) = -x^2$. In the case of the general quadratic function $f(x) = ax^2 + bx + c$, this point is given by $\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right)$. As a result, the x coordinate of the vertex is $\frac{-b}{2a}$ and the y coordinate of the vertex is $c - \frac{b^2}{4a}$. Notice that it is not necessary to memorize these formulas since they can be obtained by completing the square. Still, most people who work with parabola commit the formula $\frac{-b}{2a}$ for the x coordinate of the vertex to memory. Then they obtain the y coordinate by plugging the value $x = \frac{-b}{2a}$ into the function.

The general information is shown below.

The graph of $f(x) = ax^2 + bx + c$		
Value of *a*.	**Vertex**	**Shape**
$a > 0$	Vertex at $x = \frac{-b}{2a}$.	Parabola turning up.
$a < 0$	Vertex at $x = \frac{-b}{2a}$.	Parabola turning down.

Examples:

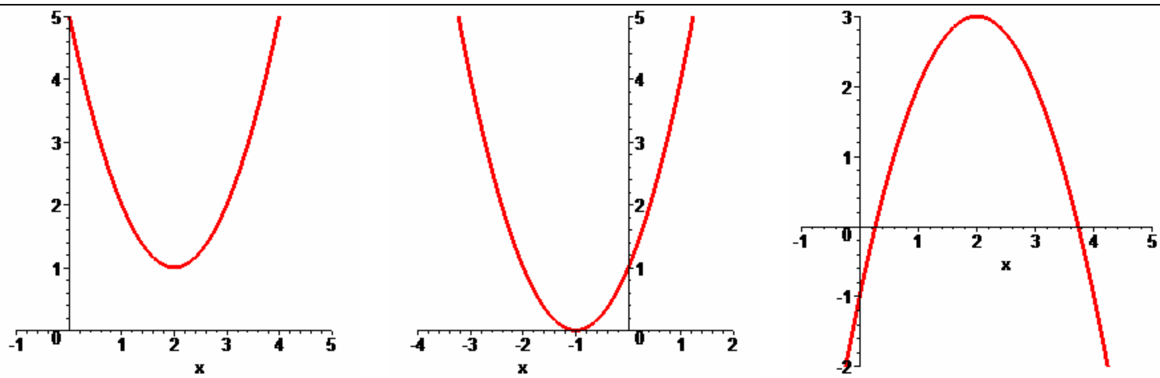
1. The graph of the function $f(x) = -x^2 + 3x$ is a **parabola that turns down** since the coefficient in from of the x^2 term is negative. From the formula above, the *x* coordinate of the vertex of the parabola is $x = \frac{-3}{2(-1)} = \frac{3}{2}$.

   The *y* coordinate of the vertex can be obtained from the formula above, or by computing $f\left(\frac{3}{2}\right) = -\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) = -\frac{9}{4} + \frac{9}{2} = \frac{9}{4}$. Consequently, the vertex of the parabola is the point $\left(\frac{3}{2}, \frac{9}{4}\right)$.

2. The graph of the function $f(x) = 2x^2 + x$ is a **parabola that turns up** since the coefficient in from of the x^2 term is positive. From the formula above, the *x* coordinate of the vertex of the parabola is $x = \frac{-1}{2(2)} = -\frac{1}{4}$.

   The *y* coordinate of the vertex can be obtained from the formula above, or by computing $f\left(-\frac{1}{4}\right) = 2\left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right) = \frac{2}{16} - \frac{1}{4} = -\frac{1}{8}$. Consequently, the vertex of the parabola is the point $\left(-\frac{1}{4}, -\frac{1}{8}\right)$.

The ***x* intercept(s)** of a quadratic function are the points (if any) where the graph crosses the *x* axis. The three quadratic functions graphed below show the possible situations.



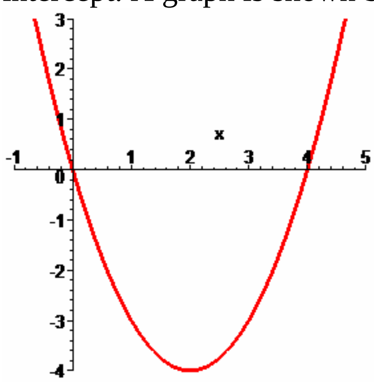
No *x* intercept. One *x* intercept Two *x* intercepts.

From the graphs above, it appears as though **the graph of a quadratic function can have either no *x* intercept, exactly one *x* intercept, or two *x* intercepts.** *y* intercepts are a little simpler. An overview of the information associated with *x* and *y* intercepts is given in the table below.

The graph of $f(x) = ax^2 + bx + c$.	
***x* intercept(s)**	***Y* intercept**
Solve $f(x) = 0$.	Evaluate $f(0)$.
i.e. solve $ax^2 + bx + c = 0$.	$f(0) = a(0)^2 + b(0) + c = c$.
The *x* intercept(s) is (are) at $(x, 0)$ where *x* solves $ax^2 + bx + c = 0$.	The *y* intercept is at $(0, c)$.

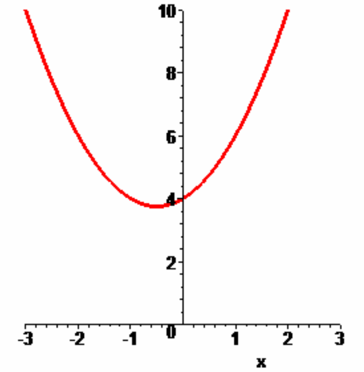
<u>Examples</u>:

1. Let $f(x) = x^2 - 4x$. The *y* intercept of the graph of *f* is at $(0, f(0))$, and $f(0) = (0)^2 - 4(0) = 0$. Consequently, the *y* intercept is $(0,0)$. The *x* intercept(s) can be found by solving $x^2 - 4x = 0$. Factoring the left hand side gives $x(x-4) = 0$. This equation tells us $x = 0$ or $x = 4$. Consequently, there are two *x* intercepts given by $(0,0)$ and $(0,4)$. Notice that in this example then point $(0,0)$ is both an *x* intercept and a *y* intercept. A graph is shown below.



2. Let $f(x) = x^2 + x + 4$. The *y* intercept of the graph of *f* is at $(0, f(0))$, and $f(0) = (0)^2 + (0) + 4 = 4$. Consequently, the *y* intercept is $(0,4)$. The *x* intercept(s) can be found by solving $x^2 + x + 4 = 0$. It is not immediately clear how to factor the left hand side. However, if we complete the square we get the equation $\left(x^2 + x + \left(\frac{1}{2}\right)^2\right) + 4 - \left(\frac{1}{2}\right)^2 = 0$, which is equivalent to

$\left(x+\frac{1}{2}\right)^2+\frac{15}{4}=0$. There is no real solution to this equation since it is impossible to add the square of a number to the positive value $\frac{15}{4}$ and get zero. Consequently, there are no *x* intercepts for this function. The graph below clearly shows that the function does not have any *x* intercepts.



A formula can be given to determine whether the graph of a quadratic function has no *x* intercept, one *x* intercept, or two *x* intercepts. Recall that the *x* intercept(s) is (are) at the point(s) $(x,0)$ where *x* solves $ax^2+bx+c=0$. We can use completing the square to show

$$\begin{aligned}0&=ax^2+bx+c\\&=a\left(x^2+\frac{b}{a}x+\Delta\right)+c-a\Delta\\&=a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+c-a\frac{b^2}{4a^2}\\&=a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}\end{aligned}$$

Consequently, the equation $ax^2+bx+c=0$ is equivalent to

$$a\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a}-c$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$ (divide by a and find a common denominator)

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$ (take the square root of both sides)

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ (subtract $\frac{b}{2a}$ from both sides and simplify)

The final equation above is referred to as **the quadratic formula** for solving the equation $ax^2 + bx + c = 0$.

The Quadratic Formula For Solving $ax^2 + bx + c = 0$.	
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Provided $b^2 - 4ac \geq 0$

The expression $b^2 - 4ac$ (under the square root above) plays an important role since we can not find the square root of a negative number. Consequently, when $b^2 - 4ac < 0$, the equation $ax^2 + bx + c = 0$ does not have a solution. When $b^2 - 4ac = 0$ the quadratic formula gives $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b \pm \sqrt{0}}{2a} = -\dfrac{b}{2a}$. So the equation $ax^2 + bx + c = 0$ has only one solution. When $b^2 - 4ac > 0$ the equation $ax^2 + bx + c = 0$ has two solutions. The importance of the term $b^2 - 4ac$ warrants a special name. We call $b^2 - 4ac$ the **discriminant** of the equation $ax^2 + bx + c = 0$. This information is summarized in the table below.

Solving $ax^2 + bx + c = 0$	
Discriminant $b^2 - 4ac$	**Solutions.**
$b^2 - 4ac < 0$	No solution
$b^2 - 4ac = 0$	One solution given by $x = -\dfrac{b}{2a}$.
$b^2 - 4ac > 0$	Two solutions given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

We can use the table above to improve our earlier table for finding intercepts.

The graph of $f(x) = ax^2 + bx + c$.	
***x* intercept(s)**	***y* intercept**
Solve $ax^2 + bx + c = 0$.	Evaluate $f(0)$.

If $b^2-4ac<0$ there is no x intercept.	$f(0)=a(0)^2+b(0)+c=c$.
If $b^2-4ac=0$ the x intercept is $\left(-\frac{b}{2a},0\right)$. If $b^2-4ac>0$ the two x intercepts are $\left(\frac{-b+\sqrt{b^2-4ac}}{2a},0\right)$ and $\left(\frac{-b-\sqrt{b^2-4ac}}{2a},0\right)$.	The y intercept is at $(0,c)$.

<u>Example</u>: We can find the vertex, the x intercept(s) and the y intercept for the graph of $f(x)=2x^2-2x-1$. The vertex has the x coordinate

$$x=-\frac{b}{2a}=-\frac{-2}{2(2)}=\frac{1}{2}$$

The y coordinate can be found by evaluating

$$f\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^2-2\left(\frac{1}{2}\right)-1=-\frac{3}{2}$$

So the vertex is the point $\left(\frac{1}{2},-\frac{3}{2}\right)$. The x intercept(s) can be found by solving the equation

$$2x^2-2x-1=0$$

The discriminant of this equation is

$$b^2-4ac=(-2)^2-4(2)(-1)=4+8=12>0$$

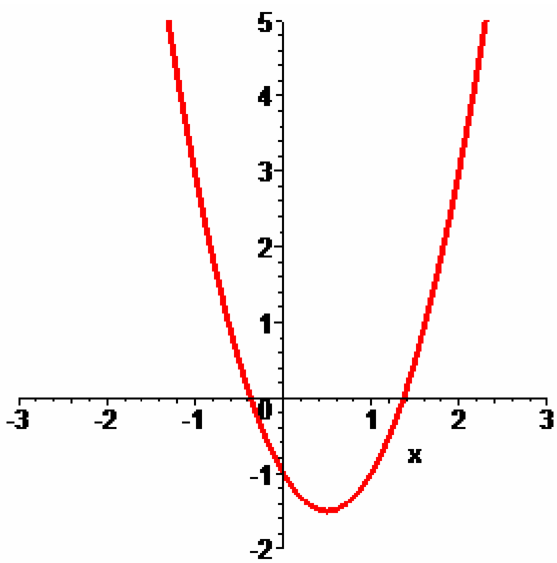
So there are two x intercepts. The quadratic formula tells us the x coordinates are given by (recall that the discriminant is 12)

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-2)\pm\sqrt{12}}{2(2)}=\frac{2\pm2\sqrt{3}}{4}=\frac{1\pm\sqrt{3}}{2}$$

So the x intercepts are

$$\left(\frac{1+\sqrt{3}}{2},0\right)\approx(2.732050808,0) \text{ and } \left(\frac{1-\sqrt{3}}{2},0\right)\approx(-.732050808,0)$$

The y intercept is the point $(0, f(0)) = (0,-1)$ The graph of $f(x) = 2x^2 - 2x - 1$ is a parabola that turns upward (since the coefficient on the x^2 term is positive). The graph is shown below.



Summary Data For $f(x)=2x^2-2x-1$
Graph Shape: Parabola turning up.
Vertex: $\left(\frac{1}{2},-\frac{3}{2}\right)$
***x* intercepts:** $\left(\frac{1+\sqrt{3}}{2},0\right)\approx(2.732050808,0)$ and $\left(\frac{1-\sqrt{3}}{2},0\right)\approx(-.732050808,0)$
***y* intercept:** $(0,-1)$

<u>Exercises</u>:

1. Use the quadratic formula to find all solutions (if any) to the equation $x^2+x-4=0$.
2. Use the quadratic formula to find all solutions (if any) to the equation $2x^2+x-7=0$.
3. Use the quadratic formula to find all solutions (if any) to the equation $x^2-2x=5$. (Hint: Bring all of the terms to one side of the equation.)
4. Use the quadratic formula to find all solutions (if any) to the equation $2x^2=3x+11$. (Hint: Bring all of the terms to one side of the equation.)
5. Sketch the parabola given by the graph of the function $f(x)=x^2-4$. Label the vertex, the *x* intercept(s) (if any), and the *y* intercept.
6. Sketch the parabola given by the graph of the function $f(x)=2x^2-4x+5$. Label the vertex, the *x* intercept(s) (if any), and the *y* intercept.
7. Sketch the parabola given by the graph of the function $f(x)=-x^2-2x-3$. Label the vertex, the *x* intercept(s) (if any), and the *y* intercept.
8. Sketch the parabola given by the graph of the function $f(x)=-3x^2+12$. Label the vertex, the *x* intercept(s) (if any), and the *y* intercept.

<u>C. Applications</u>

Quadratic functions arise in many applications. We illustrate two of these in the examples below.

Example: The ABC company manufactures camping racks for SUVs. It has fixed costs of $25,000 per month, and additional manufacturing costs of $75 for each camping rack that it produces. As a result, the ABC company has monthly costs of

$$C(x) = 25,000 + 75x$$

where x is the number of camping racks it produces in a given month. The ABC company has done some marketing analysis, and determined they can sell x of its camping racks in a given month for $175 - \frac{x}{100}$ dollars. Consequently, if the company sells x camping racks in a given month, its monthly revenue (the amount of money the company receives from sales) is given by

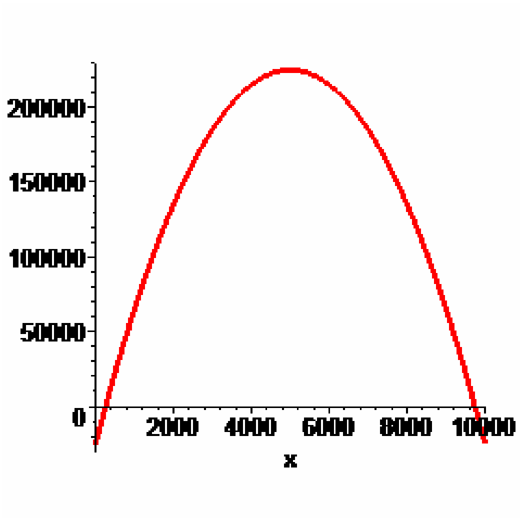
$$R(x) = (price)(number\ sold) = \left(175 - \frac{x}{100}\right)x$$

The ABC company only produces as many camping racks as it sells each month. Consequently, the monthly profit $P(x)$ is the difference between revenue $R(x)$ and cost $C(x)$. That is,

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= \left(175 - \frac{x}{100}\right)x - (25,000 + 75x) \\ &= 175x - \frac{1}{100}x^2 - 25,000 - 75x \\ &= 100x - 25,000 - \frac{1}{100}x^2 \end{aligned}$$

The **break even** point occurs where profit is zero (i.e. there is no loss and no gain). This occurs when $100x - 25,000 - \frac{1}{100}x^2 = 0$, or equivalently,

$$\frac{1}{100}x^2 - 100x + 25,000 = 0$$



From the quadratic formula, the solutions of this equation are given by $x \approx 256.58$ and $x \approx 9743.42$. The profit is plotted on the left for *x* between 0 and 10,000 units. We can see from the plot that there is a loss, and then the first break even point occurs near $x \approx 256.58$ units. The profit is maximized at the vertex of the parabola. This occurs when $x = -\frac{-100}{2/100} = 5{,}000$ units. The profit declines until the next break even point near $x \approx 9743.42$, followed by loss.

As a result, the ABC company could maximize its profit by maximizing and selling 5,000 camping racks per month at a selling price of

$$selling\ price = 175 - \frac{5{,}000}{100} = 175 - 50 = 125 \text{ dollars.}$$

At this price, the ABC company will have a monthly revenue of

$$R(5{,}000) = (price)(units\ sold) = 125(5{,}000) = 625{,}000 \text{ dollars,}$$

and their monthly cost will be

$$C(5{,}000) = 25{,}000 + 75(5{,}000) = 400{,}000$$

This will give a monthly profit of

$$P(5{,}000) = R(5{,}000) - C(5{,}000) = 625{,}000 - 400{,}000 = 225{,}000 \text{ dollars.}$$

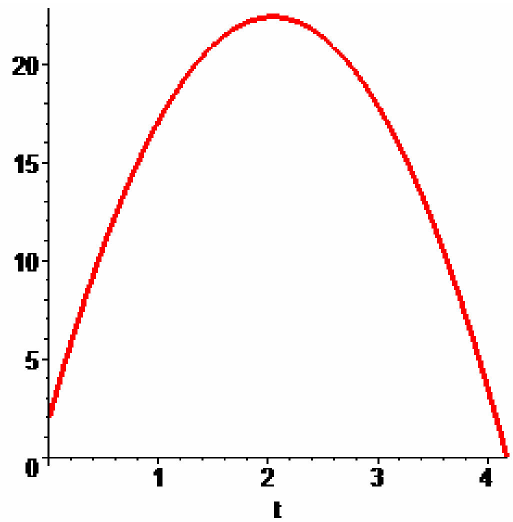
Example: The laws of physics can be used to show that if a particle is propelled into the air with an initial velocity of v_0 meters per second from a height of s_0 meters, then the position of the particle at time $t \geq 0$ is given by

$$s(t) = -4.9t^2 + v_0 t + s_0$$

until the time when the particle strikes the ground. The validity of this expression is based upon the assumption that air resistance is negligible, and the only force acting on the particle is gravity. Note that values of v_0 which are positive correspond to the particle initially moving upwards, and values of v_0 which are negative correspond to the particle initially moving downwards. Now consider a specific situation where a rock is launched upwards with an initial velocity of 20 meters per second from a height of 2 meters. According to the formula above, the height of the particle at time $t \geq 0$ is given by

$$s(t) = -4.9t^2 + 20t + 2$$

until the time when the particle strikes the ground. The particle will strike the ground when $s(t) = 0$. That is, when $-4.9t^2 + 20t + 2 = 0$. From the quadratic formula, this will occur when $t \approx 4.18$ seconds. Consequently, the formula above is valid for $0 \leq t \leq 4.18$, and from the formula, the graph of the height of the particle will be a parabola that turns down. The particle will be at its highest point at the vertex of the parabola. This occurs when $t = -\frac{20}{2(-4.9)} \approx 2.0408$ seconds, and the height of the particle at this time is given by $s(2.0408) \approx 22.408$ meters. A graph is shown below.



Exercises:

1. The Big Toe company manufactures a particular type of thong sandals. Their fixed costs are \$10,000 per month. In addition, the cost of producing their sandals is \$20 per pair. Some marketing analysis shows that they can sell x pairs of sandals each month at a price of $30 - \frac{x}{1000}$ dollars per pair. The Big Toe company only produces as many pairs of sandals as it sells. Create the cost, revenue and profit functions for the company. Can the company make a profit? If so, use the graph of the profit function to find the break even points and the number of pairs of sandals that will maximize their monthly profit. Then give the associated sales price (per pair) that will maximize profit, and give the maximum monthly profit. If not, how will things change if they can reduce their monthly fixed costs to \$5,000?
2. A particle is thrown upwards with an initial velocity of 25 meters per second from a height of 25 meters. How long will the particle be in the air? What is the maximum height that the particle reaches during its flight.