High-Dimensional Measures and Geometry

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We inspect the general result in the product space with our examples:
Examples: Fair and unfair coins: If $X = \{0,1\}, \mu(\{0\}) = 1/2 = \mu(\{1\}), f(x) = x - 1/2$ for $x \in X$, then

$$L_f(\lambda) = \ln\left(\frac{1}{2}e^{\lambda/2} + \frac{1}{2}e^{-\lambda/2}\right)$$

To compute Legendre transform, maximize $t\lambda - L_f(\lambda)$, maximum is assumed at $\lambda^* = \ln\left(\frac{1+2t}{1-2t}\right)$, $-1/2 < t \le 1/2$. [By taking derivative w. r. t. λ and equalizing to zero to solve for λ]. Thus, we have

$$L_f^*(\lambda) = -\left(\frac{1}{2}+t\right)\ln\left(\frac{1}{1+2t}\right) - \left(\frac{1}{2}-t\right)\ln\left(\frac{1}{1-2t}\right) = H\left(\frac{1}{2}-t\right) + \ln 2.$$

So, we obtain precisely what we had previously

$$\mu_n\left(\{x \in X : h(x) \le nt\}\right) \le 2^{-n}e^{nH\left(\frac{1}{2}-t\right)}.$$

Now, we let $\mu(\{0\}) = 1-p, \mu(\{1\}) = 1$ and take $f(x = x - p)$. Then,

$$L_f(\lambda) = \ln\left(pe^{\lambda(1-p)} + (1-p)e^{-p\lambda}\right) = \int e^{\lambda f} d\mu$$

and $t\lambda - L_f(\lambda)$ is maximal at

$$\lambda^* = \ln\left(\frac{(1-p)(t+p)}{p(1-p-t)}\right), \qquad -p \le t < 1-p.$$

Thus, we have

$$\begin{aligned}
L_f^*(\lambda) &= \ln\left(pe^{(1-p)\ln\left(\frac{(1-p)(t+p)}{p(1-p-t)}\right)} + (1-p)e^{-p\ln\left(\frac{(1-p)(t+p)}{p(1-p-t)}\right)}\right) \\
&= -p\ln\left(\frac{(1-p)(t+p)}{p(1-p-t)}\right) + \ln\left(e^{\ln p + \ln\left(\frac{(1-p)(t+p)}{p(1-p-t)}\right)} + e^{\ln(1-p)}\right) \\
&= -p\ln\left(\frac{(1-p)(t+p)}{p(1-p-t)}\right) + \ln\left(e^{\ln p + \ln\left(\frac{(t+p)}{p(1-p-t)}\right)}\right) + \ln(1-p) \\
&= -p\ln\left(\frac{(1-p)(t+p)}{p(1-p-t)}\right) + \ln\left(\frac{(t+p)}{(1-p-t)} + 1\right) + \ln(1-p) \\
&= -p\ln\left(\frac{(1-p)(t+p)}{p(1-p-t)}\right) + \ln\left(\frac{1}{(1-p-t)}\right) + \ln(1-p).
\end{aligned}$$

So, we have

$$\begin{aligned}
L_f^*(\lambda) &= (t+p)\ln\left(\frac{(1-p)(t+p)}{p(1-p-t)}\right) - \ln\left(\frac{1-p}{1-p-t}\right) \\
&= (t+p)\ln\left(\frac{(t+p)}{p}\right) + (t+p)\ln\left(\frac{1-p}{1-p-t}\right) - \ln\left(\frac{1-p}{1-p-t}\right) \\
&= (t+p)\ln\left(\frac{(t+p)}{p}\right) + (t+p-1)\ln\left(\frac{(1-p)}{1-p-t}\right)
\end{aligned}$$

Using Varadhan's Lemma, we have, for $h = x_1 + \cdots + x_n - np, h : X_n \to \mathbb{R}$ that

$$\begin{aligned}
\mu_n\left(\{x \in X_n : h(x) \le nt\}\right) &\le e^{-nL_f^*(t)} \\
&= \left[\left(\frac{p}{t+p}\right)^{t+p}\left(\frac{1-p}{1-p-t}\right)^{1-p-t}\right]^n \qquad (*)
\end{aligned}$$

Setting $t = \frac{\alpha\beta}{n}, p = \frac{\beta^2}{n}$, we see that the R. H. S. of $(*)$ becomes

$$\left(1 - \frac{\frac{\alpha\beta}{n}}{\frac{\alpha\beta}{n} + \frac{\beta^2}{n}}\right)^{\alpha\beta+\beta^2}\left(1 + \frac{\frac{\alpha\beta}{n}}{\frac{1-\alpha\beta}{n} - \frac{\beta^2}{n}}\right)^{n}\left(1 + \frac{\frac{\alpha\beta}{n}}{\frac{1-\alpha\beta}{n} - \frac{\beta^2}{n}}\right)^{-\alpha\beta-\beta^2}$$

which approaches to $\left[\left(1 - \frac{\alpha\beta}{\alpha\beta+\beta^2}\right)^{\alpha\beta+\beta^2} e^{\alpha\beta}\right] < 1$ as $n \to \infty$. So, this is a non-trivial asymptotic estimate.

General Insights: If X is a metric space witha probability measure μ such that $A(t) = \{x : d(x, A) \le t\}$ has measure $\mu(A(t)) \ge 1 - e^{-ct^2}$ for A with $\mu(A) \ge \frac{1}{2}$, then any 1-Lipschitz function $f : X \to \mathbb{R}$ concentrates about its median.

To see this, we consider

$$A_+ = \{x : f(x) \ge m_f\}$$

and

$$A_- = \{x : f(x) \le m_f\},$$

both of which has measure at least $1/2$. Then, we have

$$\mu(A_+(t)) \ge 1 - e^{-ct^2}, \mu(A_-(t)) \ge 1 - e^{-ct^2}$$

and

$$\mu(A_+(t) \cap A_-(t)) \ge 1 - 2e^{-ct^2}.$$

This implies that

$$\mu\left(\{s : |f(x) - m_f| \le t\}\right) \ge 1 - 2e^{-ct^2}.$$

6 Measure Concentration on High Dimensional Spheres

6.1 Gaussians as Limits of Projected Spherical Measures

We recall that we induced normalized surface measure on S^{n-1} by map $x \to \frac{x}{\|x\|}$ from γ_n. Now, we will induce measure in reverse direction.

6.1.1 Lemma. *Non-normalized Riemannian surface measure of S^{n-1} is*

$$|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

with

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx, \qquad t > 0.$$

Proof. Consider $\rho_n(x) = \frac{1}{(2\pi)^{n/2}} e^{-\|x\|^2}$, the standard Gaussian desity and let $S^{n-1}(r) = \{x \in \mathbb{R}^n : \|x\| = r\}$. Integration in polar co-ordinates gives us

$$\begin{aligned} 1 = \int_{\mathbb{R}^n} d\rho_n &= \frac{1}{(2\pi)^{n/2}} \int_0^\infty |S^{n-1}(r)| e^{-r^2/2} dr \\ &= \frac{1}{(2\pi)^{n/2}} |S^{n-1}(1)| \int_0^\infty r^{n-1} e^{-r^2/2} dr \end{aligned}$$

Substituting $t = r^2/2, rdr = dt, r^{n-2} = 2^{(n-2)/2} t^{(n-2)/2}$, we have

$$\begin{aligned} 1 &= \frac{1}{(2\pi)^{n/2}} |S^{n-1}(1)| \int_0^\infty 2^{(n-2)/2} t^{(n-2)/2} e^{-t} dt \\ &= \frac{1}{(2\pi)^{n/2}} |S^{n-1}(1)| 2^{(n-2)/2} \Gamma(n/2) \end{aligned}$$

Solving for $|S^{n-1}(1)|$, we obtain

$$|S^{n-1}(1)| = \frac{(2\pi)^{n/2} 2^{(n-2)/2}}{\Gamma(n/2)} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

□

6.1.2 Theorem. *Let $\tilde{S}^{n-1} \subset \mathbb{R}^n$ be the sphere*

$$\tilde{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = \sqrt{n}\}$$

and $\tilde{\mu}_n$ be the rotation invariant probability measure on $\tilde{S}^{n-1}$. Consider $\Phi : \tilde{S}^{n-1} \to \mathbb{R}$ be given by $\Phi(x_1, \cdots, x_n) = x_1$ and let ν_n be the induced measure on $\mathbb{R}$, i. e.,

$$\nu_n(A) = \tilde{\mu}_n(\Phi^{-1}(A)) \text{ for Borel sets } A \subset \mathbb{R}.$$

If γ_1 is the standard Gaussian measure with density $\rho_1(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, then for any Borel set $A \subset \mathbb{R}$, $\lim_{n\to\infty} \nu_n(A) = \gamma_1(A)$ and the density of ν_n converges uniformly on compact sets to ρ_1.