High-Dimensional Measures and Geometry

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6.2 Volume of a spherical cap

6.2.1 Lemma. *$B = B_a(\frac{\pi}{2} - t)$ in $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ has a measure*

$$\mu_{n+2}(B) \leq \sqrt{\frac{\pi}{8} e^{\frac{-t^2 n}{2}}}$$

Also,

$$\mu_{n+2}(B) \leq \frac{1}{2} e^{\frac{-t^2 n}{2}} (1 + \eta_n)$$

with $\eta_n \to 0$ as $n \to \infty$.

Proof. Rotate coordinates so that the center of the cap is $a = (1, 0, 0 \ldots, 0)$. If we slice the cap by hyperlanes with $\{x \in \mathbb{R}^n : x_1 = \cos\varphi\}$, then we get an n-dimensional sphere of radius $\sin\varphi$. (Assume $\varphi \geq 0$.) Integrating over φ gives

$$\mu_{n+2}(B) \frac{|\mathbb{S}^n|}{|\mathbb{S}^{n+1}|} \int_0^{\frac{\pi}{2}+t} \sin^n \varphi d\varphi$$

$$= \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{n+2}{2})}{2\pi^{\frac{n+2}{2}}} \int_t^{\frac{\pi}{2}} \cos^n \varphi d\varphi$$

Here, we have that $\cos^n \varphi \leq (e^{-\frac{\varphi^2}{2}})^n$ on $[0, \frac{\pi}{2}]$ and the stuff to the left of the integral is $\frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+1}{2})}$.

Now, letting $\varphi' = \varphi\sqrt{n}$, we have

$$\mu_{n+2}(B) \leq \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+1}{2})} \int_0^{\sqrt{n}(\frac{\pi}{2}-t)} e^{-(\varphi' + \sqrt{n}t)^2/2} d\varphi'$$

$$\leq \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+1}{2})} \frac{1}{\sqrt{n\pi}} e^{-nt^2/2} \int_0^{\infty} e^{-(\varphi')^2/2} d\varphi'$$

$$= (\frac{1}{\sqrt{2n}} \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+1}{2})} e^{-nt^2/2}$$

where the part in parenthesis $\to 1$.

By Stirling's formula, $\frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+1}{2})} \to \frac{1}{2}$ as $n \to \infty$.

It remains to show that $\frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+1}{2})\sqrt{2n}} \leq \sqrt{\frac{\pi}{8}}$

To see this, note that for $n = 1$ or $n = 3$,

$$\frac{\Gamma(2)}{\Gamma(3/2)}\frac{1}{2} = \sqrt{\frac{\pi}{8}}$$

$$\frac{\Gamma(3/2)}{\Gamma(1)}\frac{1}{\sqrt{2}} = \sqrt{\frac{\pi}{8}}$$

For the lowest dimensions, equality is achieved. We need to show that this does not get larger for higher dimensions. Using the functional equation for Γ, $\Gamma(t+1) = t\Gamma(t)$, we have

$$\frac{\Gamma(\frac{n+4}{2})}{\Gamma(\frac{n+3}{2})}\frac{1}{\sqrt{2(n+2)}} = \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{n+1}{2})}\frac{n+2}{n+1}\frac{1}{\sqrt{2n}}\frac{\sqrt{sn}}{\sqrt{2(n+2}}$$

with the factor $\frac{n+2}{n+1}\sqrt{\frac{n}{n+2}}<1$ because

$$\sqrt{1-\frac{2}{n+2}}<1-\frac{1}{n+2}$$

by Taylor expansion. □

We get the immediate consequence:

6.2.2 Theorem. *If $f : \mathbb{S}^{n+1} \to \mathbb{R}$ is $1 - Lipschitz$ and m_f is its median, then for $\varepsilon>0$ we have:*

$$\mu_{n+1}(\{x \in \mathbb{S}^{n+1} : |f(x) - m_f| \geq \varepsilon\}) \leq \sqrt{\frac{\pi}{2}}e^{-\varepsilon^2 n/2}$$

Proof. We follow the general principle/insight:

Define $A_+ = \{x \in \mathbb{S}^{n+1} : f(x) \geq m_f\}$ and $A_- = \{x \in \mathbb{S}^{n+1} : f(x) \leq m_f\}$. Then $\mu(A_+)$,$\mu(A_-) \geq \frac{1}{2}$, and using Levy's theorem,

$$\mu_{n+2}(\{x : d(x, A_\pm) \leq \varepsilon\{) \geq \mu_{n+2}(\{x : d(x, B_a) \leq \varepsilon\})$$

$$\geq 1 - \sqrt{\frac{\pi}{8}}e^{-\varepsilon^2 n/2}$$

Thus intersecting the two sets gives by union bound:

$$\mu_{n+2}(\{x : |f(x) - m_f| \leq \varepsilon\}) \geq 1 - \sqrt{\frac{\pi}{2}}e^{-\varepsilon^2 n/2}$$

□

6.3 Concentration for Gaussian measures

Consider $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ for some large N, and let $\Phi : (x_1, x_2, \ldots, x_N) \mapsto \sqrt{N}(x_1, x_2, \ldots, x_n)$ be the "scaled" projection onto the first n coordinates. Then, as $N \to \infty$, μ_N induces a measure which converges to γ_n.

Next, we deduce measure concentration for γ_n from that of μ_N.

6.3.3 Theorem. *(Borell) Let $A \subset \mathbb{R}^n$ be closed and $t \geq 0$, and let $H = \{x \in \mathbb{R}^n : x \cdot \alpha \leq b\}$ for fixed $\alpha \in \mathbb{R}^n$,$b \in \mathbb{R}$ so that $\gamma_n(H) = \gamma_n(A)$, and then,*

$$\gamma_n(\{x : d(x, A) \leq t\}) \geq \gamma_n(\{x : d(x, H) \leq t\})$$

Before proving this theorem, we note that if $H = \{x \in \mathbb{R}^n : x \cdot \alpha \geq 0\}$, then $\gamma_n(H) = \frac{1}{2}$. Also, denoting $H(t) = \{x \in \mathbb{R}^n : d(x, H) \leq t\}$, then

$$\gamma_n(H(t)) = \frac{1}{\sqrt{2n}} \int_{-\infty}^{t} e^{-x^2/2} dx$$

where, without loss of generality, we have $\alpha = (1, 0, 0, \ldots, 0)$.

We estimate the measure of $H(t)$.

6.3.4 Lemma. *For $H(t) \subset \mathbb{R}^n$ as above, $t \geq 0$, $\gamma_n(H(t)) \geq 1 - e^{-t^2/2}$.*

Proof. We have by Laplace transform,

$$\gamma_n(\{x : x_1 > t\}) \leq e^{-\lambda t} E[e^{\lambda x_1}] = e^{-\lambda t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\lambda x_1} e^{-x_1^2/2} dx_1$$

$$= e^{-\lambda t} e^{\lambda^2/2}$$

Optimizing λ gives $\lambda = t$, so $\gamma_n(\{x : x_1 \leq t\}) \geq 1 - e^{-t^2/2}$.

□

As before, we deduce a concentration result.

6.3.5 Theorem. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be $1 - Lipschitz$ and m_f be its median with respect to γ_n. Then, for $\varepsilon > 0$,*

$$\gamma_n(\{x : |f(x) - m_f| \geq \varepsilon\}) \leq 2e^{-\varepsilon^2/2}$$

Proof. The proof of this theorem follows the same general strategy as before and is omitted. □

We now prove Borell's theorem.

Proof. Given $A \subset \mathbb{R}^n$, A bounded and closed, choose $H = \{x \in \mathbb{R}^n : x_1 \leq b\}$ such that $\gamma_n(A) = \gamma_n(H)$. Then we know that

$$\lim_{n \to \infty} \mu_N(\Phi^{-1}(A)) = \lim_{N \to \infty} \mu_N(\Phi^{-1}(H)) = \gamma_n(H)$$

Also, $\Phi^{-1}(H)$ is a spherical cap with apex $a = (1, 0, 0, \ldots, 0)$ and (geodesic) radius $r_N = \arccos(\frac{n}{\sqrt{N}})$, where N is a scaling term.

Now consider $A(t) = \{x : d(x, A) \leq t\}$ and $H(t) = \{x : x_1 \leq b + t\}$.

Then $\Phi^{-1}(H(t)) = B_a(r_N + \varepsilon_N)$ with a radius $r_N + \varepsilon_N = \arccos(\frac{b+t}{\sqrt{N}})$

We see that:

$$\frac{t}{\sqrt{N}} \leq \varepsilon_N \leq \frac{t}{\sqrt{N}}(1 + \frac{(b+t)^2/N}{1-(b+t)^2/N})^{\frac{1}{2}}$$
$$\leq \frac{t}{\sqrt{N}} + C\frac{(b+t)^2}{N^{3/2}}$$

and the volume is

$$\mu_N(B_a(r_n + \varepsilon_N)) = \frac{|\mathbb{S}^{N-2}|}{|\mathbb{S}^{N-1}|}\int_{-1}^{(b+t)\sqrt{N}} (1 - x_1^2)^{N-2}(1 + \frac{x_1^2}{1-x_1^2})^{\frac{1}{2}} dx_1$$

Considering $A(t)$, by boundedness, $\exists c''{>}0$ such that

$$|x_j| \leq \frac{c''}{\sqrt{N}}, 1 \leq j \leq N$$

and

$$x \in \Phi^{-1}(A(t)),$$

so $\|(x_1, \ldots, x_n)\|^2 \leq \frac{n(c'')^2}{N}$.

The measure of $\Phi^{-1}(A(t))$ is $\mu_N(\Phi^{-1}(A(t))$

$$= |\mathbb{S}^{N-n-1}||S^{N-1}|\int_{A(t)/\sqrt{N}} (1 - \|x\}^2)^{N-n-1}(1 + \frac{\|x\|^2}{1-\|x\|^2})^{n/2} dx_1 dx_2 \ldots dx_n$$
$$\leq CN^{n/2}$$

Consider $E_N = \Phi^{-1}(A) \subset \mathbb{S}^{N-1}$.

Then $\Phi^{-1}(A(t)) \subset E_N(B_N)$ with

$$\frac{t}{\sqrt{N}} \leq B_N \leq \frac{t}{\sqrt{N}}(1 + \frac{n(c'')^2/N}{1-n(c'')^2/N})^{\frac{1}{2}} \leq t\sqrt{N} + \frac{c'''}{N^{3/2}}$$

Thus, by optimality of spherical caps,

$$\mu_n(B_a(r_N + \beta_n)) \leq \mu_N(E_N(\beta_N)) + \eta_n$$

where $\eta_N \to 0$ as $n \to \infty$.

Taking $N \to \infty$ gives

$$\lim_N \mu_N(B_a(r + \beta_N)) = \gamma_N(H(t)) \leq \lim_N \mu_N(E_N(\beta_N)) = \gamma_n(A(t))$$

Convergence on the left hand side is because $\beta_N - \varepsilon_N {<} \frac{C}{N^{3/2}}$ and changes in the (upper) limit of the integral smaller than $C/N^{3/2}$ do not contribue to the volume, since $|\mathbb{S}^{N-1}|/|\mathbb{S}^{N-1}| \leq c'N^{\frac{1}{2}}$.

Convergence on the right hand side follows from a similar argument concerning the volume of $E_N(B_N)$ versus $\Phi^{-1}(A(t))$.

□

To generalize to unbounded A, take limits over bounded subsets.

This concludes the section on concentration about the median.

7 Concentration about the mean for Gaussians

Maurey/Pisier idea: Instad of m_f,∂_f, let the function concentrate about "itself".

Take $f : \mathbb{R}^n \to R$. Define $F : \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$, $F(x,y) = f(x) - f(y)$ in the space with measure $\gamma_{2n} = \gamma_n(x) \times \gamma_n(y)$.

If F concentrates near 0, then f concentrates somewhere.

We will also use the rotation-invariance of the Gaussian measure under

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Instead of Lipschitz continuity, consider the smaller set of differentiable functions with $\| \bigtriangledown f \| = (\sum_{i=1}^n (\frac{\partial f}{\partial x_i})^2)^{\frac{1}{2}} \leq 1$.

We let $E[f] = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-\|x\|^2/2} dx$ and note that $e^{a \cdot x}$ gives $E[e^{a \cdot x}] = e^{\|a\|^2/2}$.

7.0.1 Theorem. *Consider a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ with $E[f] = 0$. Then*

$$E[e^f] \leq E[\exp(\frac{\pi^2}{8} \| \bigtriangledown f \|^2)]$$

if the right hand side is finite.

Proof. We introduce $F : \mathbb{R}^{2n} \to \mathbb{R}$, $F(x,y) = f(x) - f(y)$, then $E[e^f] \leq E[e^F]$ because

$$E[e^F] = \int_{\mathbb{R}^{2n}} e^{f(x)-f(y)} d\gamma_{2n} = \int_{\mathbb{R}^n} e^{f(x)} d\gamma_n(x) \int_{\mathbb{R}^n} e^{-f(y)} d\gamma_n(y)$$

$$\geq E[e^f]$$

where

$$\int_{\mathbb{R}^n} e^{-f(y)} d\gamma_n(y) \geq e^{-\int_{\mathbb{R}^n} f(y) d\gamma_n(y)}$$

Now, rotate coordinates and let $G(x,y,\theta) = f(x\cos\theta + y\sin\theta)$ with $x(\theta) = x\cos\theta + y\sin\theta$, $y(\theta) = \cos\theta y - \sin\theta x = x'(\theta)$.

Then,

$$G(x,y,0) = f(x), G(x,y,\frac{\pi}{2}) = f(y)$$

and

$$F(x,y) = G(x,y,0) - G(x,y,\frac{\pi}{2}) = \int_{\frac{\pi}{2}}^{0} \frac{\partial}{\partial\theta} G(x,y,\theta) d\theta.$$

□