0.0.1 Theorem. If \( f : \mathbb{R} \to \mathbb{R} \) is differentiable with \( E[f] = 0 \), then \( E[f] \leq E\left[\exp\left(\frac{\pi^2}{8}||\nabla f||^2\right)\right] \), assuming this RHS is finite.

Proof.

\[
F(x, y) := f(x) - f(y)
\]

\[
E[f] \leq E[F] = E\left[\exp\left(-\int_0^{\pi/2} \frac{\partial}{\partial \theta} G(x, y, \theta) \, d\theta\right)\right] = E\left[\exp\left(-\int_0^{\pi/2} \nabla f(x(\theta)) \cdot x'(\theta) \, d\theta\right)\right]
\]

with \( G(x, y, \theta) = f(x(\theta)) = f(x \cos \theta, y \cos \theta) \), and \( x'(\theta) = -x \sin \theta + y \cos \theta = y(\theta) \), which looks like rotation.

\[
E[F] = E\left[\exp\left(-\frac{2}{\pi} \int_0^{\pi/2} \frac{\partial}{\partial \theta} G(x, y, \theta) \, d\theta\right)\right] \leq \frac{2}{\pi} \int_0^{\pi/2} E\left[\exp\left(-\frac{\pi}{2} \nabla f(x(\theta)) \cdot x'(\theta)\right)\right] \, d\theta.
\]

Fix \( \theta \), and change variables, \( \tilde{x}_j = x_j \cos \theta + y_j \sin \theta \), \( \tilde{y}_j = -x_j \sin \theta + y_j \cos \theta \). Then,

\[
E[g(x, y)] = E[g(\tilde{x}, \tilde{y})],
\]

invariant under rotation. Thus by invariance of measure,

\[
E[F] \leq \frac{2}{\pi} \int_0^{\pi/2} E\left[\exp\left(-\frac{\pi}{2} \nabla f(x) \cdot y\right)\right] \, d\theta = \frac{2}{\pi} \int_0^{\pi/2} E\left[\exp\left(\frac{\pi^2}{4}||\nabla(x)||^2/2\right)\right] \, d\theta = E\left[\exp\left(\frac{\pi^2}{4}||\nabla(x)||^2/2\right)\right].
\]

\[\square\]

0.0.2 Corollary. If \( f : \mathbb{R} \to \mathbb{R} \) is differentiable, with \( ||\nabla f|| \leq 1 \), and \( a := E[f] \), then \( \gamma_n(\{x \in \mathbb{R}^n; |f(x) - a| \leq t\}) \geq 1 - 2e^{-\frac{t^2}{\pi a}} \)
Proof. Let $\lambda \geq 0$, use Laplace transform method,
\[
\gamma_n \{x; f(x) - a \geq t\} \leq e^{-\lambda t} E \left[ e^{\lambda(f-a)} \right] \leq e^{-\lambda t} e^{\lambda^2 \pi^2/8},
\]
choosing $\lambda = \frac{4t}{\pi^2}$ gives
\[
\gamma \{x; |f(x) - a| \leq t\} \geq 1 - 2e^{-\frac{2t^2}{\pi^2}}.
\]

0.0.3 Question. Can we use result for $\gamma_n$, and $\Phi = x \mapsto \frac{x}{||x||}$, to obtain concentration about the mean on spheres?

0.0.4 Idea. Extend differentiable 1-Lipschitz functions $f : S^{n-1} \to \mathbb{R}$ to $\mathbb{R}^n$, consider $g(x) = ||x||f(\frac{x}{||x||})$. Then $g$ is differentiable when $f$ is (except at $x = 0$).

\[
Du g(x) = (Du ||x||) f \left( \frac{x}{||x||} \right) + ||x|| Du f \left( \frac{x}{||x||} \right)
\]
\[
|Du g(x)| \leq (1)||f||_{\infty} + \max_{||u||=1, u \cdot x = 0} ||x|| |Du f \left( \frac{x}{||x||} \right)| \leq ||f||_{\infty} + 1.
\]
Now, if $\int f \, d\mu_n = 0$, then $||f||_{\infty} \leq \frac{\pi}{2}$ because
\[
|f(x) - 0| = |f(x) - \int_{S^{n-1}} f(x') \, d\mu_n(x')| = |\int_{S^{n-1}} (f(x) - f(y)) \, d\mu_n(y)| \leq \int_{S^{n-1}} d(x, y) \, d\mu_n(y) = \frac{\pi}{2}.
\]
As a consequence, if $f$ is differentiable on $S^{n-1}$ and 1-Lipschitz, then $g$ as defined above is differentiable on $\mathbb{R}^n \setminus \{0\}$, and is 3-lipschitz.