High-Dimensional Measures and Geometry

Lecture Notes from Mar 2, 2010

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Proof. Consider the extention, g, of f to all of $\mathbb{R}^n$, Let $G(x,y) = g(x) - g(y)$, then

$$E_{\mu_n \times \mu_n}\left[e^{f(x)-f(y)}\right] = E_{\tilde{\gamma}_{2n}}\left[e^{G(x,y)}\right],$$

$\tilde{\gamma}_{2n}$ with density $\frac{1}{(2\pi\sigma^2)^n} e^{\frac{-||x||^2}{2\sigma^2}}$, where σ is chosen appropriately, '(so sum of random variables variance $= 1$)'. Then,

$$E_{\tilde{\gamma}_{2n}}\left[e^{G(x,y)}\right] \leq E_{\tilde{\gamma}_{2n}}\left[e^{\frac{\pi^2}{8}}||\nabla f||^2\sigma^2\right]$$

so, if $||\nabla g|| \leq 3$, then we conclude that

$$E_{\mu_n}\left[e^{\lambda f}\right] \leq e^{9\pi^2(\lambda\sigma)^2/8},$$

we know that as $n \to \infty$, $n\sigma^2 \to 1$. Using the Laplace transform method,

$$\mu_n\left(\{x \in S^{n-1}; f(x) \geq t\}\right) \leq e^{-\lambda t} E\left[e^{\lambda f}\right] \leq e^{-t\lambda}\exp(9\pi^2(\lambda\sigma)^2/8) \Rightarrow$$

$$\mu\left(\{x \in S^{n-1}; f(x) \geq t\}\right) \leq e^{\frac{-2t^2}{9\pi^2\sigma^2}}$$

$\square$

0.0.1 Question. How much smaller is the set of differentiable functions with $||\nabla f||^2 \leq 1$ compared to 1-Lipschitz functions?

0.0.2 Answer. Smaller by "ϵ". Prove that any 1-Lipschitx function can be approximated uniformly by differentiable functions.

0.0.3 Theorem. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be 1-Lipschitz, define the localized averages $f(x) = \frac{1}{|B_\epsilon|(x)} \int_{B_\epsilon|(x)} g(y)\,dy$, where $B_\epsilon|(x) = \{y \in \mathbb{R}^n; ||x - y \leq \epsilon\}$. Then f is differentiable, and for all $x \in \mathbb{R}$, $|f(x)\ g(x)| \leq \frac{\epsilon n}{n+1} \leq \epsilon$, and $||\nabla f|| = 1$.*

Proof. First, check $n = 1$, then

$$f(x) = \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} g(y)\,dy$$

and by the fundamental theorem of calculus, f is differentiable:

$$f'(x) = \frac{1}{2\epsilon}\left[g(x+\epsilon) - g(x-\epsilon)\right], |f'(x)| = \frac{1}{2\epsilon}|g(x+\epsilon) - g(x-\epsilon)| \le \frac{1}{2\epsilon}(2\epsilon) = 1.$$

Moreover,

$$|f(c) - g(x)| = |g(x) - \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} g(y)\,dy| = |\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}(g(x) - g(y))\,dy| \le \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon}|g(x) - g(y)|\,dy = \frac{\epsilon^2}{2\epsilon} = \frac{\epsilon}{2}.$$

In higher dimensions, similar analysis works. We only prove that $D_u f(x) = \nabla f(c)\cdot u, ||u|| = 1$ gives $|D_u f(x| \le 1$. Without loss of generality, $x = 0$, $u = (1, 0, ..., 0)$. Have $D_u f(0) = \frac{d}{dt} f(tu)|_{t=0}$. Define the disk $D_\epsilon = \{x \in \mathbb{R}^n; x \perp u, ||x|| \le \epsilon\}$. We compute $f(x)$ in cylindrical coordinates,

$$f(tu) = \frac{1}{|B_\epsilon|}\int_{B_\epsilon + t_u} g(z)\,dz = \frac{1}{|B_\epsilon|}\int_{D_\epsilon}\int_{t-\sqrt{\epsilon^2 - ||y||^2}}^{t+\sqrt{\epsilon^2 - ||y||^2}} g(y + se_1)\,ds\,dy,$$

where $e_1 = (1, 0, 0, .)$, so

$$\frac{d}{dt} f(tu)|_{t=0} = \frac{1}{|B_\epsilon|}\left|\int_{D_\epsilon}\left(g(y + \sqrt{\epsilon^2 + ||y||^2}e_1) - g(y + \sqrt{\epsilon^2 - ||y||^2}e_1)\right)\,dy\right|$$

$$\le \frac{1}{|B_\epsilon|}\int_{D_\epsilon} 2\sqrt{\epsilon^2 - ||y||^2}\,dy = \frac{|B_\epsilon|}{|B_\epsilon|} = 1 \Rightarrow$$

$$|D_\epsilon f| \le 1,\ \forall\, u \in \mathrm{Ball}(\mathbb{R}^n)$$

.

Moreover,

$$|f(0) - g(0)| = \frac{1}{|B_\epsilon|}\left|\int_{D_\epsilon}(g(y) - g(0))\right| \le \frac{1}{|B_\epsilon|}\int_{D_\epsilon}|g(x) - g(0)||\,dy = \frac{|S^{n-1}|}{|B_\epsilon|}\int_0^\epsilon r^n\,dr$$

$$= \frac{\epsilon^{n+1}}{n+1}\frac{|S^{n-1}|}{\epsilon^n|B_1|} = \frac{\epsilon}{n+1}\frac{|S^{n-1}|}{|B_1|} = \frac{n\epsilon}{n+1}$$

□

Back to concentration, we now know that f concnetrates on a set of large measure, but where? Given an ϵ, and 1-Lipschitz function $f : S^{n-1} \to \mathbb{R}$, then there exists a subspace $V \subset \mathbb{R}^n$, $\dim(V)$ linear in n, s.t. $f_{V\cap S^{n-1}}$ is $\epsilon-$close to a constant. Note, S^{n-1} is $(n-1)$-sphere, $S^{n-1} \cap V$ is again a unit sphere of dimension $\dim(V) - 1$.

0.0.4 Theorem. *There exists a universal constant, $\kappa > 0$, s.t. $\epsilon > 0, \forall n \in \mathbb{N}$, any 1-Lipschitz function $f : S^{n-1} \to \mathbb{R}$ there exists a constant, C (e.g. median or averge) and a subspace $V \subset \mathbb{R}^n$, s.t. $|f(x) - C| \le \epsilon \forall x \in V \cap S^{n-1}$ and $dim(V) \ge \frac{\kappa\epsilon^2}{\ln(1/\epsilon)} n$.*

To prove this, we begin with a lemma about equidistributed points on the sphere

0.0.5 Lemma. *Given n-dimensional vector space W, with norm $||\cdot||$ and $\Sigma = \{x \in W; ||x|| = 1\}$, then $\forall \delta > 0, \exists S \subset \Sigma$ with*

1) $\forall x \in S,\ \inf\{||y - x||; y \in A, y \neq x\} \leq \delta$

2) $|S| \leq (1 + \frac{2}{\delta})^n$

The same is true for appropriate sets in $\{x \in W; ||x|| = 1\}$.