High-Dimensional Measures and Geometry

Lecture Notes from March 11, 2010

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6.2.1 Theorem. *Let $n, N, 0 < \delta < \frac{1}{2}$ be given. For $V \in G_n(\mathbb{R}^N)$ we denote by P_v the orthogonal projection on to V. There exist constants c_1, c_2 depending on δ such that for $k \leq \frac{c_1 n}{\ln\left(\frac{N}{k}\right)}$,*

$$(*) \qquad (1-\delta)^2 \|x\|^2 \leq \frac{N}{n}\|P_v x\|^2 \leq \frac{1}{(1-\delta)^2}\|x\|^2$$

for all $x \in \cup_l W_l$, $W_l = span\{e_{j_1}(l), e_{j_2}(l), \dots e_{j_k}(l)\}$ and V can be chosen from a set of measure,

$$\mu_{N,n}(\{V : (*) \ holds\}) \geq 1 - 2e^{-c_2 n}$$

Proof. By lemma, $(*)$ holds for all $x \in W$ with fixed $W = span\{e_{j_1}, e_{j_2}, \dots e_{j_k}\}$. There are ${}^N C_k$ such subspaces and by Stirling's bound,

$${}^N C_k \leq \left(\frac{eN}{k}\right)^k$$

Using the union bound, we get that probability for $(*)$ to fail for atleast one choice of W is bounded above by,

$$\begin{aligned} & 2\left(\frac{eN}{k}\right)^k \left(1+\frac{8}{\delta}\right)^k e^{-\alpha\left(\frac{\delta}{2}\right)^2 n} \\ = \quad & 2e^{-\alpha\left(\frac{\delta}{2}\right)^2 n + k\ln\left(1+\frac{8}{\delta}\right) + k\ln\left(\frac{eN}{k}\right)} \end{aligned}$$

If c_1 is fixed then by choosing $k \leq \frac{c_1 n}{\ln\left(\frac{N}{k}\right)}$ makes the exponent bounded by $-c_2 n$ if

$$c_2 \leq \alpha\left(\frac{\delta}{2}\right)^2 - c_1 - c_1\frac{\ln\left(1+\frac{8}{\delta}\right)}{\ln\left(\frac{N}{k}\right)} = \alpha\left(\frac{\delta}{2}\right)^2 - c_1\left(1 + \frac{\ln\left(1+\frac{8}{\delta}\right)}{\ln\left(\frac{N}{k}\right)}\right)$$

Thus if c_1 is small enough then $c_2 > 0$. $\square$

6.3 Consequences of the Restricted Isometry Principle

6.3.2 Definition. *Given an $m \times n$ matrix $\Phi, m < n, s \in \mathbb{N}$ then the Restricted Isometry Principle constant δ_s is defined to be the smallest number for which,*

$$(1-\delta_s)\|x\|^2 \leq \|\Phi x\|^2 \leq (1+\delta_s)\|x\|^2$$

for all s-sparse x, i.e. $x \in span\{e_{j_1}, e_{j_2}, \ldots e_{j_s}\}$.

6.3.3 Problem. Given $y = \Phi x \in \mathbb{R}^m$, recover all s-sparse x from "measurement" y.

Strategy: Minimize $\|x\|_1$ subject to $\Phi x = y$.

6.3.4 Theorem. *Given an $m \times n$ matrix $\Phi, m < n, s \in \mathbb{N}$, with Restricted Isometry Principle constant $\delta_{2s} < \sqrt{2} - 1$, then l_1-minimization recovers x from $y = \Phi x$.*

The same strategy also works approximately for noisy data. Given $y = \Phi x + z$ with $\|z\| < \epsilon$, minimize $\|x\|_1$ subject to $\|y - \Phi x\| \leq \epsilon$.

6.3.5 Theorem. *Given Φ as above with $\delta_{2s} < \sqrt{2} - 1, \|z\| \leq \epsilon$, then there exists $c_1 > 0$ such that the above l_1-minimization gives a solution x^* for which $\|x^* - x\| \leq c_1 \epsilon$.*

6.3.6 Lemma. *If $x \in span\{e_{j_1}, e_{j_2}, \ldots e_{j_s}\}$ and $x' \in span\{e_{j'_1}, e_{j'_2}, \ldots e_{j'_{s'}}\}$ and $j_l \neq j_{l'}$ for any $1 \leq l \leq s$ and $1 \leq l' \leq s'$, then $|\langle \Phi x, \Phi x' \rangle| \leq \delta_{s+s'} \|x\| \|x'\|$.*

Proof. If x, x' are unit vectors with "disjoint support" as assumed then,

$$2(1-\delta_{s+s'}) \leq \|\Phi(x+x')\|^2 \leq 2(1+\delta_{s+s'})$$

Now using parallelogram identity,

$$\|\Phi(x+x')\| = \frac{1}{4}\left| \|\Phi(x+x')\|^2 + \|\Phi(x-x')\|^2 \right| \leq \delta_{s+s'}$$

$\square$

Proof. (of noisy reconstruction theorem)
Observe that if x^* is a minimizer to l_1-norm in the set $\{\tilde{x} : \|\Phi\tilde{x} - y\| \leq \epsilon\}$ then the triangle inquality gives,

$$(0) \qquad \|\Phi(x^* - x)\| \leq \underbrace{\|\Phi x^* - y\|}_{\leq \epsilon} + \|\underbrace{y - \Phi x}_{z}\| \leq 2\epsilon$$

Now consider $x^* = x + h$ and show that h is small enough. Let $h = h_0 + h_1 \ldots$, each h_i being s-sparse and
h_0 being supported on the support of x,
h_1 being supported on the set of s largest coefficients of h on the complement of the support of x,
h_2 contains the next s largest coefficients,

$\vdots$

We bound $\|\sum_{i=2} h_i\|$ by $\|h_0 + h_1\|$. To this end, we note

$$\|h_j\| \leq \sqrt{s}\|h_j\|_\infty \leq \frac{1}{\sqrt{s}}\|h_{j-1}\|_1$$

and thus,

$$\begin{aligned} \sum_{j\geq 2}\|h_j\| &\leq s^{-\frac{1}{2}}(\|h_1\| + \|h_2\| + \ldots) \\ &= s^{-\frac{1}{2}}\|h - h_0\|_1 \end{aligned}$$

Also,

$$\|h - h_0 - h_1\| = \|\sum_{j\geq 2} h_j\| \leq s^{-\frac{1}{2}}\|h - h_0\|_1 \tag{1}$$

Next we bound,

$$\begin{aligned} \|x\|_1 &\geq \|x + h\|_1 \\ &= \|x_0 + h\|_1 + \|h - h_0\|_1 \\ &\geq \|x\|_1 - \|h_0\|_1 + \|h - h_0\|_1 \end{aligned}$$

This gives,

$$\|h - h_0\|_1 \leq \|h_0\|_1 \tag{2}$$

Applying inequality (1) and (2) with $\|h_0\|_1 \leq s^{\frac{1}{2}}\|h_0\|$ gives,

$$\|h - h_0 - h_1\| \leq \|h_0\|$$

Now we consider $\|h_0 + h_1\|$. We have,

$$\begin{aligned} |\langle \Phi(h_0 + h_1), \Phi(h)\rangle| &\leq \underbrace{\|\Phi(h_0 + h_1)\|}_{\leq\sqrt{1+\delta_{2s}}\|h_0+h_1\|}\|\Phi(\underbrace{\overbrace{h}}_{\leq 2\epsilon}{}_{x - x*})\| \\ &\leq 2\epsilon\sqrt{1 + \delta_{2s}}\|h_0 + h_1\| \end{aligned}$$

Also from lemma,

$$|\langle \Phi h_0, \Phi h_j\rangle| \leq \delta_{2s}\|h_0\|\|h_j\|$$

and the same inequality holds when h_0 is replaced by h_1. So since h_0 and h_1 have disjoint support,

$$\begin{aligned} \|h_0\| + \|h_1\| &\leq \sqrt{2}\|h_0 + h_1\| \\ (1 - 2\delta_{2s})\|h_0 + h_1\|^2 &\leq \|\Phi(h_0 + h_1)\|^2 \\ &= |\langle \Phi(h_0 + h_1), \Phi(\underbrace{\overbrace{h_0 + h_1}^{}}_{-\sum_{j\geq 2} h_j}{}_{h + h_0 + h_1 - h})\rangle| \\ &\leq \|h_0 + h_1\|\left(2\epsilon\sqrt{1 + 2\delta_{2s}} + \sqrt{2}\delta_{2s}\sum_{j\geq 2}\|h_j\|\right) \end{aligned}$$

Now from, $\sum_{j\geq 2} \|h_j\| \leq s^{-\frac{1}{2}}\|h - h_0\|$,

$$\|h_0 + h_1\| \leq \alpha\epsilon + \rho s^{-\frac{1}{2}}\|h - h_0\|$$

with $\alpha = \frac{2\sqrt{1+\delta_{2s}}}{1-2\delta_{2s}}$ and $\rho = \frac{\sqrt{2}\delta_{2s}}{1-2\delta_{2s}}$.
This means,

$$\begin{aligned} & \|h_0 + h_1\| \leq \alpha\epsilon + \rho\|h_0 + h_1\| \\ \implies \quad & \|h_0 + h_1\| \leq \frac{1}{1-\rho}\alpha\epsilon \end{aligned}$$

Finally,

$$\begin{aligned} \|h\| &\leq \|h_0 + h_1\| + \|h - h_0 - h_1\| \\ &\leq 2\|h_0 + h_1\| \\ &\leq 2\,\underbrace{\frac{1}{1-\rho}}_{c}\alpha\,\epsilon \end{aligned}$$

□