High-Dimensional Measures and Geometry

Lecture Notes from March 30 and April 1, 2010

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9.3.1 Remark. The constant in the exponent is not as "good" (meaning large) as in the previous proof of concentration on the sphere, but our technique is more general. In particular, if we have a norm on $\mathbb{R}^n$ and define strict convexity by requiring that any $\|x\| = \|y\| = 1$, $\|x-y\| \geq \varepsilon$ satisfy $\|\frac{x+y}{2}\| \leq 1-\delta$, then this proof works in verbatim fashion, i.e. it gives concentration on "spheres" of l^p spaces, $1<p<\infty$.

9.4 Concentration for strictly log-concave measures

We study more uses of the Prékopa-Leindler inequality. Next we (re-)derive concentration for gaussian measures and other, strictly log-concave measures. As usual, we begin with the Laplace transform.

9.4.2 Theorem. *If μ is a probability measure on $\mathbb{R}^n$ with a nowhere vanishing density $\rho = e^{-u}$, $u; \mathbb{R}^n \to \mathbb{R}$ and $\exists c>0$ so that*

$$\frac{u(x)+u(y)}{2} - u\frac{x+y}{2} \geq \frac{c}{2}\|x-y\|^2$$

then if A is closed and bounded, then

$$\int e^{c(d(x,A))^2} d\mu(x) \leq \frac{1}{\mu(A)}$$

Proof. Define f, g, h as follows:

$f(x) = \exp(c(d(A,x))^2 - u(x))$
$g(x) = \chi_A(x)e^{-u(x)}$
$h(x) = e^{-u(x)}$

By assumption,

$$h(\frac{x_y}{2}) = e^{-u((x+y)/2)} \geq e^{-\frac{1}{2}(u(x)+u(y))+\frac{c}{2}\|x-y\|^2}$$

We check two cases:

i) $y \notin A$, then $g(y) = 0$, so $h((x-y)/2) \geq f^{\frac{1}{2}}(x)g^{\frac{1}{2}}(y) = 0$
ii) $y \in A$, then $\|x-y\| \geq d(x,A)$

Thus,

$$h(\frac{x+y}{2} \geq e^{-\frac{1}{2}(u(x)+u(y))+\frac{c}{2}(d(x,A))^2}$$

if $y \in A$, where the right side of the inequality is equal to $f^{\frac{1}{2}}(x)g^{\frac{1}{2}}(y)$.

Now, applying the Prékopa-Leindler inequality,

$$\int_{\mathbb{R}}^{n} h(x)dx \geq (\int_{\mathbb{R}}^{n} f(x)dx)^{\frac{1}{2}}(\int_{\mathbb{R}}^{n} g(x)dx)^{\frac{1}{2}}$$

where the term inside the first integral equals $e^{-u} = \mu(\mathbb{R}^n) = 1$, the term $f(x)dx = e^{c(d(x,A))^2}d\mu(x)$, and $g(x)dx = \chi_A(x)d\mu(x)$. This gives $1 \geq (\int_{\mathbb{R}}^{n} e^{c(d(x,A))^2}d\mu(x))^{\frac{1}{2}})\mu(A))^{\frac{1}{2}}$.

Now, squaring both sides and rearranging gives the desired inequality.

9.4.3 Corollary. *If given a probability measure μ on $\mathbb{R}^n$ with density $\rho = e^{-u}$ as in the preceding theorem, $t \geq 0$, and $A \subset \mathbb{R}^n$ with $\mu(A) \geq \frac{1}{2}$, then*

$$\mu(\{x \in \mathbb{R}^n : d(x, A) \geq t\}) \leq 2e^{-ct^2}$$

Proof. The Laplace transform method gives

$$\mu(\{x \in \mathbb{R}^n : d(x, A) \geq t^2\}) \leq e^{-ct^2} \int_{\mathbb{R}}^{n} e^{c(d(x,A))^2}d\mu \leq 2e^{-ct^2}$$

where $\int_{\mathbb{R}}^{n} e^{c(d(x,A))^2}d\mu \leq 2$.

9.4.4 Example. If $u(x) = \frac{\|x\|^2}{2} + \frac{n}{2}\ln(2\pi)$ then μ is the standard gaussian measure in n dimensions, and $\frac{u(x)+u(y)}{2} - u(\frac{x+y}{2}) = \frac{\|x\|^2}{4} + \frac{\|y\|^2}{4} - \frac{\|x+y\|^2}{8} = \frac{\|x-y\|^2}{8}$ so we can choose $c = \frac{1}{4}$ in the above corollary. This gives us that

$$\gamma_n(\{x \in \mathbb{R}^n : d(x, A) \geq t\}) \leq 2e^{-t^2/4}$$

for $\gamma_n(A) \geq \frac{1}{2}$.

Compare with the better numerical constant for our earlier results.

9.5 Concentration for log-concave measures

Can we adapt to the situation without strict log-concavity?

Idea: Instead of a "neighborhood" of a set A, consider tA, $t>1$. We use scaling for $\mu(\alpha A + \beta B) \geq \mu^{\alpha}(A)\mu^{\beta}(B)$, $B - \mathbb{R}^n \backslash tA$ and appropriate α,β so that $\alpha A + \beta B \subset \mathbb{R}^n \backslash A$, and then $1 - \mu(A) \geq \mu^{\alpha}(A)\mu^{\beta}(B) \Rightarrow \mu(b) \leq (\mu(A))^{-\frac{\alpha}{\beta}}(1 - \mu(A))^{\frac{1}{\beta}}$.

9.5.5 Theorem. Let μ be a log-concave measure on $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$, convex, symmetric, and compact, i.e. $A = \bar{A}$, $A = -A$, and A is bounded.

Then for $t>1$,$\mu(\mathbb{R}^n \backslash (tA)) \leq \mu(A)^{\frac{1-t}{2}}(1 - \mu(A))^{\frac{t+1}{2}}$.

Proof. Choose $B = \mathbb{R}^n \backslash (tA)$, and let $\alpha = \frac{t-1}{t+1}$ and $\beta = \frac{2}{t+1}$, and then

$$\mu(\alpha A + \beta B) \geq \mu^{\alpha}(A)\mu^{\beta}(B)$$

. We claim that $\alpha A + \beta B \subset \mathbb{R}^n \backslash A$.

Choose $a \in A$ and $b \in B$. and assume that $\exists c \in A$ so that $\alpha a + \beta b = c$, so then

$$b = \frac{1}{\beta}(c - \alpha a) = \frac{t+1}{2}c - \frac{t-1}{2}a$$

We would like to make this a convex combination, so we take instead that this is equal to $\frac{t+1}{2}c + \frac{t-1}{2}(-a) \in A$, since $c \in A$ and $-a \in A$ (by the fact that A is symmetric).

But this contradicts the fact that $b \in B$ and $B \cap A = \emptyset$.

Thus, $\mu(A^c) \geq \mu^{\alpha}(A)\mu^{\beta}(B)$, and so

$$\mu(B) \leq \mu^{-\frac{\alpha}{\beta}}(A)(1-\mu(A))^{\frac{1}{\beta}} = \mu^{-\frac{t-1}{2}}(A)(1-\mu(A))^{\frac{t+1}{2}}$$

9.5.6 Example. If $\mu(A) = \frac{2}{3}$, then $\mu(\mathbb{R}^n \backslash tA) \leq \frac{2}{3}2^{-(t+1)/2}$, but if $\mu(A) = \frac{1}{2}$, then $\mu(\mathbb{R}^n \backslash tA) \leq \frac{1}{2}$.

So, we get that the "rate" of concentration depends on the measure, which is a qualitatively new idea.

9.5.7 Example. Let $d\mu(x) = \frac{1}{2}e^{-|x|}dx$ on $\mathbb{R}$. Choose $A = [-a, a] \Rightarrow \mu(A) = 1 - e^{-a}$

Then $\mu(tA) = 1 - e^{-at}$

$\mu(\mathbb{R} \backslash tA) = e^{-at}$

and thus the measure decreases exponentially but with rate depending on $\mu(A)$.

We apply the preceding corollary to normed spaces.

9.5.8 Corollary. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a norm, μ a log-concave measure on $\mathbb{R}^n$, $r > \frac{1}{2}$, let $\rho \geq 0$ so that $\mu(\{x \in \mathbb{R}^n : p(x) \leq \rho\}) = r$, then for all $t > 1$, $\mu(\{x \in \mathbb{R}^n : p(x) > t\rho\}) \leq r^{\frac{1-t}{2}}(1-r)$.

Proof. Let $A = \{x \in \mathbb{R}^n : p(x) \leq \rho\}$.

Recall Hölder's inequality.

If $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq \infty$, and measure μ, f, g so that fg is integrable, then

$$|\int fg d\mu| \leq (\int_X |f|^p d\mu)^{\frac{1}{p}} (\int_X |g|^q d\mu)^{\frac{1}{q}}$$

If μ is a probability measure and if f is integrable, then

$$\int |f| d\mu \leq (\int |f|^p d\mu)^{\frac{1}{p}} = \|f\|_p$$

More generally, if f is q-integrable and $q \leq p$, then

$$\|f\|_q \leq \|f\|_p$$

If $f \geq 0$ and $\ln f$ is integrable, then we can take the endpoint derivative at $p = 1$ and get

$$\frac{d}{dp}|_{p=1}(\int |f| d\mu)^p \leq \frac{d}{dp}|_{p=1} \int |f|^p d\mu$$

$$\Rightarrow \|f\|_1 \ln \|f\|_1 \leq \int |f| \ln |f| d\mu$$

9.5.9 Theorem. If μ is a log-concave probability measure on $\mathbb{R}^n$ and $p : \mathbb{R}^n \to \mathbb{R}$ a norm, then $\exists c>0$ so that

$$(\int p^q d\mu)^{\frac{1}{q}} \leq cq \int p d\mu$$

for all $q \geq 1$.

Proof. Assume that $\int p d\mu = 1$. Otherwise, rescale p.

Choose $\rho>2$. Since $p \geq 0$, we have that $\mu(\{x \in \mathbb{R}^n : p(x) \geq \rho\}) \leq \frac{1}{\rho}$, by taking complements, we have that $\mu(\{x \in \mathbb{R}^n : p(x) \leq \rho\}) \geq \frac{\rho-1}{\rho}$.

Thus, we get from the preceding corollary that

$$\mu(\{x \in \mathbb{R}^n : p(x) \geq \rho t\}) \leq \mu^{\frac{1-t}{2}}(A)(1-\mu(A))^{\frac{1+t}{2}}$$

$$\leq (\frac{\rho-1}{\rho})^{\frac{1-t}{2}}(\frac{1}{\rho})^{\frac{1+t}{2}}$$

$$= \frac{1}{\rho}(\rho-1)^{\frac{1-t}{2}}$$

where $A = \{x \in \mathbb{R}^n : p(x) \leq \rho\}$.

(We note in making this calculation that $p(\frac{x}{t}) \geq \rho$.)

Choose $\rho = 1 + e<4$. Then we get,

$$\mu(\{x \in \mathbb{R}^n : p(x) \geq 4t\})$$

$$\leq \frac{e^{\frac{1}{2}}}{1+e}e^{-t/2} \leq e^{-t/2}$$

Define the cumulative distribution function

$$F(t) = \mu(\{x \in \mathbb{R}^n : p(x) \leq t\})$$

and then

$$1 - F(t) \leq e^{-t/8}$$

where $t>4$.

Now,

$$\int p^q d\mu = \int_0^\infty t^q dF(t)$$

$$= -\int_0^\infty t^q d(1-F(t))$$

$$= -t^q(1-F(t))|_0^\infty + \int_0^\infty qt^{q-1}(1-F(t))dt$$

$$\leq \int_0^4 qt^{q-1}(1-F(t))dt + \int_0^\infty qt^{q-1}e^{-t/8}dt$$

$$
\begin{aligned}
&= 4^q + q8^{q-1}\int_4^\infty (\frac{t}{8})^{q-1}e^{-t/8}dt \\
&\leq 4^q + q8^{q-1}(8)\int_0^\infty (\frac{t}{8})^{q-1}e^{-t/q}dt = 4^q + q8^q\Gamma(q)
\end{aligned}
$$