9.5.1 Lemma. Given a norm \( p : \mathbb{R}^n \to \mathbb{R} \), let \( B = \{ x \in \mathbb{R}^n : p(x) \leq 1 \} \). Let \( \delta > 0, \rho > 1 \) such that \( \mu(\rho B) \geq (1 + \rho)\mu(B) \) then for some \( c > 0 \) we have,

\[
\mu(tB) \leq ct\mu(B)
\]

for all \( 0 < t < 1 \).

Proof. (continued from the previous class)
Let \( \kappa(m) \) be such that,

\[
\frac{\kappa(m)}{m} = \frac{\mu(\frac{1}{2m}B)}{\mu(B)}
\]

For all \( m \in \mathbb{N} \) we have proved,

\[
\mu(t_mB) \leq \frac{4\delta}{\rho} t_m\mu(B) \quad \forall t_m = \frac{1}{2m}
\]

So consider all \( m \in \mathbb{N} \) such that \( \kappa(m) > \frac{2\delta}{\rho} \), thus,

\[
\mu\left(\frac{1}{2m}B\right) > \frac{2\delta}{\rho} \frac{1}{m}\mu(B) > \frac{\delta}{\rho m}\mu(B)
\]

We recall the Brunn-Minkowski inequality

\[
\mu(A_{\lambda\tau}) \geq \mu^\lambda(A_\tau)\mu^{1-\lambda}\left(\frac{1}{2m}B\right)
\]

and choosing \( \tau = \tau', \lambda = \frac{1}{\tau'}, \) and using assumptions on \( m \in \mathbb{N}, \mu\left(\frac{1}{2m}B\right) > \frac{\delta\mu(B)}{\rho m} \) gives,

\[
\tau(A_1) \geq \frac{\delta\mu(B)}{\rho m}
\]

Repeating the above procedure for \( A_{1-\frac{1}{m}}, A_{1-\frac{2}{m}}, \ldots A_{1-\frac{i}{m}} \) with \( \tau = \tau', \lambda = \frac{i}{\tau m} \) to get estimate for \( A_{\frac{1}{m}} \) gives,

\[
\mu\left(A_{\frac{1}{m}}\right) \geq \frac{\mu(B)}{m} \left(\frac{\delta}{\rho}\right)^{1-\frac{i}{m}} \frac{\kappa}{m}
\]
So adding these contributions and using \( \bigcup_{A_i} \subseteq B \), with \( A_0 = \frac{1}{2m} B \), yields,

\[
\mu(B) \geq \sum_{i=0}^{m-1} \mu(A_i)
\]

and,

\[
\mu(B) \geq \frac{\mu(B) \delta}{m} \frac{1 - \frac{\rho \kappa}{\delta}}{\rho \left( \frac{\rho \kappa}{\delta} \right)^{\frac{1}{m}}}
\]

By our assumption \( \kappa \geq \frac{2\delta}{\rho} \), so

\[
1 \geq \frac{\kappa \delta}{m \left( 1 - \left( \frac{\rho \kappa}{\delta} \right)^{\frac{1}{m}} \right)} = \frac{\kappa \left( 1 - \frac{\delta}{\rho \kappa} \right)}{m \left( \frac{\rho \kappa}{\delta} \right)^{\frac{1}{m}} - 1} \geq \frac{\kappa}{m \left( \frac{\rho \kappa}{\delta} \right)^{\frac{1}{m}} - 1}
\]

So,

\[
\kappa \leq 2m \left( \left( \frac{\rho \kappa}{\delta} \right)^{\frac{1}{m}} - 1 \right)
\]

Now, the sequence \( \kappa(m) \) must be bounded because for each \( m \geq 2 \),

\[
\kappa \leq \frac{2}{\epsilon} \left( \left( \frac{\rho \kappa}{\delta} \right)^{\epsilon} - 1 \right)
\]

is a difference quotient for \( x \mapsto \left( \frac{\rho \kappa}{\delta} \right)^{x} \), \( 0 \leq x \leq \frac{1}{2} \) and thus the difference quotient is bounded. By derivative at right endpoint \( x = \frac{1}{2} \) and thus,

\[
\kappa \leq 2 \left( \frac{\rho \kappa}{\delta} \right)^{\frac{1}{2}} \ln \left( \frac{\rho \kappa}{\delta} \right)
\]

\[
\kappa^{\frac{1}{2}} \leq 2 \left( \frac{\rho}{\delta} \right)^{\frac{1}{2}} \ln \left( \frac{\rho \kappa}{\delta} \right)
\]

So \( \kappa \) cannot grow arbitrarily large.

We claimed

\[
\ln \left( \int_{\mathbb{R}^n} pd\mu \right) \leq c + \int_{\mathbb{R}^n} \ln pd\mu
\]

Assume \( \int_{\mathbb{R}^n} pd\mu = 1 \). We want to show \( \int \ln pd\mu > -\infty \). If \( t > 0 \), let \( B_t = \{ x \in \mathbb{R}^n : p(x) \leq t \} \) pick radius such that \( \mu(B_t) = \frac{2}{3} \). Since \( \mu \) is a probability measure,

\[
\mu(\{ x \in \mathbb{R}^n : p(x) \geq 4 \}) \leq \int_{\mathbb{R}^n} \frac{p(x)}{4} d\mu = \frac{1}{4}
\]
So $\rho \leq 4$.
Using norm concentration for log concave measures with $r = \frac{2}{3}, t = 3$,

$$\mu(B_{3\rho}) \geq 1 - \frac{2}{3} \cdot \frac{1}{4} = \frac{5}{6}$$

Define norm $p' = \frac{p}{\rho}$, then

$$B' = \{ x \in \mathbb{R}^n : p'(x) \leq 1 \}$$
$$= \{ x \in \mathbb{R}^n : p(x) \leq \rho \} = B_{\rho}$$

and $\mu(B') \geq (1 + \delta) \mu(B_3)$ with $\delta > 0$. Thus for some $\epsilon > 0$, $\mu(B_t) \leq ct$, for all $0 \leq t \leq \epsilon$ we select $\epsilon < 1$.

Let $F(t) = \mu(B_t)$ then,

$$\int_{\mathbb{R}^n} \ln p d\mu = \int_0^{\infty} \ln t dF(t)$$
$$\geq \int_0^1 \ln t dF(t)$$
$$= \left( \ln t \right) F(t) \bigg|_0^1 - \int_0^1 \frac{1}{t} F(t) dt$$

$$= - \int_0^\epsilon \frac{1}{t} F(t) dt - \int_\epsilon^1 \frac{1}{t} F(t) dt$$
$$\geq -c\epsilon - \int_0^1 \frac{1}{t} F(t) dt$$
$$\geq -c\epsilon - \frac{1}{\epsilon} > -\infty$$