High-Dimensional Measures and Geometry

Lecture Notes from April 8, 2010

taken by Anando Sen

9.5.1 Lemma. *Given a norm* $p : \mathbb{R}^n \to \mathbb{R}$, *let* $B = \{x \in \mathbb{R}^n : p(x) \leq 1\}$. *Let* $\delta > 0, \rho > 1$ *such that* $\mu(\rho B) \geq (1+\rho)\mu(B)$ *then for some* $c > 0$ *we have,*

$$\mu(tB) \leq ct\mu(B)$$

for all $0 < t < 1$.

Proof. (continued from the previous class)
Let $\kappa(m)$ be such that,

$$\frac{\kappa(m)}{m} = \frac{\mu(\frac{1}{2m}B)}{\mu(B)}$$

For all $m \in \mathbb{N}$ we have proved,

$$\mu(t_m B) \leq \underbrace{\frac{4\delta}{\rho}}_{c} t_m \mu(B) \quad \forall t_m = \frac{1}{2m}$$

So consider all $m \in \mathbb{N}$ such that $\kappa(m) > \frac{2\delta}{\rho}$,thus,

$$\mu\left(\frac{1}{2m}B\right) > \frac{2\delta}{\rho}\frac{1}{m}\mu(B) > \frac{\delta}{\rho m}\mu(B)$$

We recall the Brunn-Minkowski inequality

$$\mu(A_{\lambda\tau}) \geq \mu^{\lambda}(A_{\tau})\mu^{1-\lambda}\left(\frac{1}{2m}B\right)$$

and choosing $\tau = \tau'$, $\lambda = \frac{1}{\tau'}$, and using assumptions on $m \in \mathbb{N}$, $\mu\left(\frac{1}{2m}B\right) > \frac{\delta\mu(B)}{\rho m}$ gives,

$$\tau(A_1) \geq \frac{\delta\mu(B)}{\rho m}$$

Repeating the above procedure for $A_{1-\frac{1}{m}}, A_{1-\frac{2}{m}}, \ldots A_{\frac{t}{m}}$ with $\tau = \tau'$, $\lambda = \frac{i}{m\tau'}$ to get estimate for $A_{\frac{i}{m}}$ gives,

$$\mu\left(A_{\frac{i}{m}}\right) \geq \frac{\mu(B)}{m}\left(\frac{\delta}{\rho}\right)^{1-\frac{i}{m}}\kappa^{\frac{i}{m}}$$

So adding these contributions and using $\bigcup_{A_{\frac{i}{m}}}^{m-1} \subseteq B$, with $A_0 = \frac{1}{2m}B$, yields,

$$\mu(B) \geq \sum_{i=0}^{m-1} \mu(A_{\frac{i}{m}})$$

and,

$$\mu(B) \geq \frac{\mu(B)}{m}\frac{\delta}{\rho}\frac{1-\frac{\rho\kappa}{\delta}}{1-\left(\frac{\rho\kappa}{\delta}\right)^{\frac{1}{m}}}$$

By our assumption $\kappa \geq \frac{2\delta}{\rho}$, so

$$\begin{aligned} 1 &\geq \frac{\kappa}{m}\frac{\frac{\delta}{\rho k}-1}{1-\left(\frac{\rho\kappa}{\delta}\right)^{\frac{1}{m}}} \\ &= \frac{\kappa}{m}\frac{1-\frac{\delta}{\rho k}}{\left(\frac{\rho\kappa}{\delta}\right)^{\frac{1}{m}}-1} \\ &\geq \frac{\kappa}{m}\frac{\frac{1}{2}}{\left(\frac{\rho\kappa}{\delta}\right)^{\frac{1}{m}}-1} \end{aligned}$$

So,

$$\kappa \leq 2m\left(\left(\frac{\rho\kappa}{\delta}\right)^{\frac{1}{m}-1}\right)$$

Now, the sequence $\kappa(m)$ must be bounded because for each $m \geq 2$,

$$\kappa \leq \frac{2}{\epsilon}\left(\left(\frac{\rho\kappa}{\delta}\right)^{\epsilon}-1\right)$$

is a difference quotient for $x \mapsto \left(\frac{\rho\kappa}{\delta}\right)^x$, $0 \leq x \leq \frac{1}{2}$ and thus the difference quotient is bounded. By derevative at right endpoint $x = \frac{1}{2}$ and thus,

$$\kappa \leq 2\left(\frac{\rho x}{\delta}\right)^{\frac{1}{2}} \ln\left(\frac{\rho x}{\delta}\right)$$

$$\kappa^{\frac{1}{2}} \leq 2\left(\frac{\rho}{\delta}\right)^{\frac{1}{2}} \ln\left(\frac{\rho x}{\delta}\right)$$

So κ cannot grow arbitrarily large.
We claimed

$$\ln\left(\int_{\mathbb{R}^n} p d\mu\right) \leq c + \int_{\mathbb{R}^n} \ln p d\mu$$

Assume $\int_{\mathbb{R}^n} p d\mu = 1$. We want to show $\int \ln p d\mu > -\infty$. If $t > 0$, let $B_t = \{x \in \mathbb{R}^n : p(x) \leq t\}$ pick radius such that $\mu(B_\rho) = \frac{2}{3}$. Since μ is a probability measure,

$$\mu(\{x \in \mathbb{R}^n : p(x) \geq 4\}) \leq \int_{\mathbb{R}^n} \frac{p(x)}{4} d\mu = \frac{1}{4}$$

So $\rho \leq 4$.
Using norm concentration forfor log concave measures with $r = \frac{2}{3}, t = 3$,

$$\mu(B_{3\rho}) \geq 1 - \frac{2}{3}.\frac{1}{4} = \frac{5}{6}$$

Define norm $p' = \frac{p}{\rho}$, then

$$\begin{aligned} B' &= \{x \in \mathbb{R}^n : p'(x) \leq 1\} \\ &= \{x \in \mathbb{R}^n : p(x) \leq \rho\} = B_\rho \end{aligned}$$

and $\underbrace{\mu(B')}_{\frac{2}{3}} \geq (1+\delta)\underbrace{\mu(B'_3)}_{\geq \frac{5}{6}}$ with $\delta > 0$. Thus for some $\epsilon > 0$, $\mu(B_t) \leq ct$, for all $0 \leq t \leq \epsilon$ we select $\epsilon < 1$.
Let $F(t) = \mu(B_t)$ then,

$$\begin{aligned} \int_{\mathbb{R}^n} \ln p d\mu &= \int_0^\infty \ln t dF(t) \\ &\geq \int_0^1 \ln t dF(t) \\ &= \underbrace{(\ln t)F(t)|_0^1}_{0} - \int_0^1 \frac{1}{t} F(t) dt \\ &= -\int_0^\epsilon \frac{1}{t} F(t) dt - \int_\epsilon^1 \underbrace{\frac{1}{t}}_{\leq \frac{1}{\epsilon}} \underbrace{F(t)}_{\leq 1} dt \\ &\geq -c\epsilon - \int_0^1 \frac{1}{\epsilon} \underbrace{F(t)}_{\leq 1} dt \\ &\geq -c\epsilon - \frac{1}{\epsilon} > -\infty \end{aligned}$$

□