High-Dimensional Measures and Geometry

Lecture Notes from April 22, 2010

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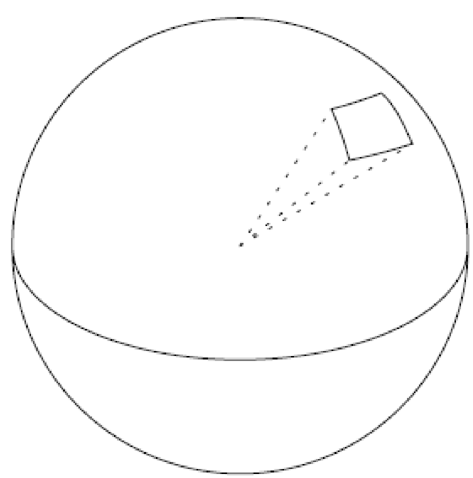
We start this lecture by finding the volume of closed unit ball in $\mathbb{R}^n$. Then we take the ball with volume one, which has a pretty big radius, and project the mass distribution to the $n-1$ dimensional slices of a fixed direction. We see that that the projected distribution forms approximately the Gaussian distribution. In the second part we will consider polytopes and we state some relations between the number of faces of the polytope and its distance the the closed unit ball. This presentation is a summary of Section 1 and 2 of [1]. Let

$$B_2^n = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \leq 1\}.$$

That is, B_2^n is the closed unit ball with respect to l_2 norm. So let v_n represent this volume. We will use the following generalized fact which is well known when n is 2 or 3. If a cone in $\mathbb{R}^n$ has height h and has base staying in an $n-1$ dimensional subsape with volume B then the volume of the cone is given by Bh/n. We will approximate the volume v_n by considering the ball B_2^n as the union of cones. The surface will be the base of the cones. A realization of this technique for $n = 3$ is given in the figure. The region selected almost acts as a cone with height approximately 1.



We will obtain the "surface area" of the closed unit ball in terms of its volume. So let, as usual,

$$S^{n-1} = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 = 1\}.$$

Then the surface area of S^{n-1} is nv_n. To find the volume v_n of B_2^n we will use the spherical polar integration. For a point x in $\mathbb{R}^n$ let $\theta = x/\|x\|_2$ and $r = \|x\|_2$. So we can write $x = r\theta$ with θ being on the unit sphere S^{n-1}. For an integrable function $f : \mathbb{R}^n \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} f dx = \int_{r=0}^{\infty} \int_{S^{n-1}} f(r\theta) d\theta r^{n-1} dr.$$

Here the factor r^{n-1} appears because the surface area of the sphere of radius r is r^{n-1} times the surface area of the unit sphere. Note that if $d\theta$ is the area measure on S^{n-1} then the normalized Lebesgue measure σ is given by $nv_n d\sigma = d\theta$. Thus we have

$$\int_{\mathbb{R}^n} f dx = nv_n \int_{r=0}^{\infty} \int_{S^{n-1}} f(r\theta) d\sigma(\theta) r^{n-1} dr.$$

Now we will integrate the function $f(x) = e^{-\|x\|^2/2} = \exp(-\frac{1}{2}(\sum x_i^2))$ both with respect to the the cartesian coordinates and the spherical polar coordinates.

$$\int_{\mathbb{R}^n} f dx = \int_{\mathbb{R}^n} \prod e^{-x_i^2/2} dx = \prod \int_{\mathbb{R}^n} e^{-x_i^2/2} dx = (\sqrt{2\pi})^n.$$

On the other hand

$$\int_{\mathbb{R}^n} f dx = nv_n \int_{r=0}^{\infty} \int_{S^{n-1}} e^{-r^2/2} r^{n-1} d\sigma(\theta) dr = nv_n \int_{r=0}^{\infty} e^{-r^2/2} r^{n-1} dr = v_n 2^{n/2} \Gamma(\frac{n}{2}+1).$$

So we get

$$v_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}.$$

It is interesting that the volume v_n of B_2^n is extremely small for large n. In fact by using the Stirling formula

$$\Gamma(\frac{n}{2}+1) \approx \sqrt{2\pi} e^{-n/2} \left(\frac{n}{2}\right)^{(n+1)/2}$$

we have that, for large n,

$$v_n \approx \left(\sqrt{\frac{2\pi e}{n}}\right)^n.$$

For example to obtain the ball of volume 1, the radius must be roughly

$$r = \sqrt{\frac{n}{2\pi e}}.$$

A ball of radius r in $\mathbb{R}^n$ has a volume $v_n r^n$. So to obtain the ball of volume 1 the radius has to be

$$r = v_n^{-1/n}.$$

We will consider the slices of this ball and compute volume of this $n-1$ dimensional balls. A slice with a distance to the origin x has radius $\sqrt{r^2-x^2}$ so its volume is

$$v_{n-1}\left(\sqrt{r^2-x^2}\right)^{n-1}=v_{n-1}r^{n-1}\left(1-\frac{x^2}{r^2}\right)^{\frac{n-1}{2}}.$$

Note that the expression $v_{n-1}r^{n-1}$ is the volume of the $n-1$ ball with radius $v_n^{-1/n}$. Applying the Stirling formula it is not hard to show that it approximates $\sqrt{e}$ for large n. Also since r is roughly $\sqrt{n/2\pi e}$ we see that volume of the slice having distance x to the origin is

$$\sqrt{e}\left(1-\frac{x^2}{r^2}\right)^{\frac{n-1}{2}}=\sqrt{e}\left(1-\frac{2\pi e x^2}{n}\right)^{\frac{n-1}{2}}\approx\sqrt{e}\exp(-\pi e x^2)=e^{-\pi e x^2+1/2}.$$

Note that this is Gaussian distribution with variance $1/(2\pi e)$. Consequently the projection of the mass distribution of the n-ball of volume 1 approximately forms a Gaussian distribution on the $n-1$ dimensional subspace. Also it is intersting to see that the variance is independent of n.

In this part we consider the symmetric polytopes and state a relation between the number of facets and the distance of the polytope to the unit ball. Let K and L be two convex bodies in $\mathbb{R}^n$. The distance $d(K,L)$ between K and L is defined to be the least positive number d such that there is a linear image $\tilde{L}$ of L with $\tilde{L}\subset K\subset d\tilde{L}$. Note that d is not a metric, it is symmetric and multiplicative. For any K $d(K,K)=1$. Infact $\log d$ is a metric.

11.0.1 Theorem. *Let K be a symmetric polytope in $\mathbb{R}^n$ with $d(K,B_2^n)$. Then K has at least exp$\big(n/(2d^2)\big)$ facets. For each n there is a polytope with 4^n facets whose distance form the ball is at most 2.*

A symmetric polytope in $\mathbb{R}^n$ with m pairs of facets is a slice (through the center) of a cube in $\mathbb{R}^m$. To see this note that the polytope must be intersection of m slabs in $\mathbb{R}^n$ each of them is given by $\{x:|\langle x,v_i\rangle|\le 1\}$ for some non-zero vector v_i. So the polytope is given by

$$\{x:|\langle x,v_i\rangle|\le 1 \text{ for } i=1,...,m\}.$$

Define $T:\mathbb{R}^n\to\mathbb{R}^m$ by $Tx=(\langle x,v_1\rangle,...,\langle x,v_m\rangle)$. The image of $\mathbb{R}^n$ under T is an n dimensional subspace of $\mathbb{R}^m$, say H, such that its intersection with $[-1,1]^m$ is exactly the image of the polytope under T. Conversely any n dimensional slice of $[-1,1]^m$ is a polytope with at most m pair of faces.

We close this part by stating an upper and a lower bound for the spherical cups in $\mathbb{R}^n$. As usual $S^{n-1}=\{x\in\mathbb{R}^n:\|x\|=1\}$. For a unit vector v and $0\le\epsilon<1$ we define

$$C(\epsilon,v)=\{u\in S^{n-1}:\ \langle u,v\rangle\ge\epsilon\}.$$

This set forms a spherical cup. We use the notation $C_r(v)$ for the spherical cup of radius r centered at v. Note the cup $C(\epsilon,v)$ is not defined by the radius. We will use the the normalized Lebesgue on the surface.

11.0.2 Lemma. *The cup $C(\epsilon,v)$ has measure at most $e^{-n\epsilon^2/2}$.*

11.0.3 Lemma. *For $0 \leq r \leq 2$, $C_r(v)$ has measure at least $\frac{1}{2}(r/2)^{n-1}$.*

Reference:

Ball, Keith. *An Elementary Introduction to Modern Convex Geometry*. Flavors of Geometry MSRI Publications Volume **31**, (1997).