10.2 Products Of Graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. By product of the graphs $G_1$ and $G_2$, denoted by $G_1 \times G_2$, we mean the graph having the vertex set $V_1 \times V_2$ and the edge set defined as follows:

$$(u_1, u_2; v_1, v_2) \text{ with } u_1, v_1 \in V_1 \text{ and } u_2, v_2 \in V_2 \text{ is an edge of the product graph } G_1 \times G_2$$

$$u_1 = v_1 \text{ and } (u_2, v_2) \in E_2 \text{ or } u_2 = v_2 \text{ and } (u_1, v_1) \in E_1.$$  

**Example:** Consider $I_1 = (\{0, 1\}, (0, 1))$. Then $I_1 \times I_1$ will be the graph with vertex set $V = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and the edge set

$$E(I_1 \times I_1) = \{(0, 0; 0, 1), (1, 0; 1, 1), (0, 0; 1, 0), (0, 1; 1, 1)\}$$

which directly follows from the definition. Note that $I_1 \times I_1$ is same as $I_2$. In general we have

$$I_n \times I_m = I_{n+m}.$$  

The Laplacian $\triangle$ of the product graph $G_1 \times G_2$ given by

$$\triangle = \triangle_1 \otimes I_{V_2} + I_{V_1} \otimes \triangle_2$$

where $\triangle_1$ and $\triangle_2$ are the Laplacian of $G_1$ and $G_2$.

**Example:** Consider the previous example where

$$\triangle_1 = \triangle_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$  

Then

$$\triangle = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}.$$
is the $\triangle_2$, Laplacian of $I_2$.

Since $\triangle_1 \otimes I_{V_2}$ and $I_{V_1} \otimes \triangle_2$ commute, the eigenvalues of $\triangle$ is the sum of eigenvalues of $\triangle_1$ and $\triangle_2$. So the smallest non-zero eigenvalue of $\triangle$ is given by

$$\lambda_1(\triangle) = \min\{\lambda_1(\triangle_1), \lambda_1(\triangle_2)\}.$$  

We can iterate this tensoration and for example we can obtain the Laplacian $\triangle_n$ of $I_n$

$$\lambda_1(\triangle_n) = \lambda_1(\triangle) = \left(\frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}}\right)^2\right) = 2.$$

This completes the explanation why the equality holds for $I_n$ in preceding theorem.

We have already established that

$$\lambda_1(\triangle) \leq \frac{|V|}{|V| - 1}d$$

where $d$ is the minimal degree. For $I_n$, $d = n$ so this bound does not contain enough information. We will construct a better estimation.

**10.2.1 Theorem.** Let $G = (V, E)$ be a connected graph and $A, B \subset V$ be two disjoint subsets with distance $\rho = d(A, B) = \min\{d(u, v) : u \in A, v \in B\}$.

Let $E(A)$ be the set of edges with both end points in $A$. Similarly define $E(B)$. Then

$$|E| - |E(A)| - |E(B)| \geq \lambda_1(\triangle)\rho^2 \frac{|A||B|}{|A| + |B|}.$$  

**Proof.** Recall that if $f \in V^\mathbb{R}$ with $\sum f(v) = 0$ then $\langle \triangle f, f \rangle \geq \lambda_1(\triangle)\|f\|^2$. We will apply this result to a special function $f$. Let

$$a = \frac{|A|}{|V|} \text{ and } b = \frac{|B|}{|V|}$$

and consider the function $g$ defined by

$$g(v) = \frac{1}{a} - \frac{1}{\rho} \left(\frac{1}{a} + \frac{1}{b}\right) \min\{d(v, A), \rho\}.$$

Note that if $v \in A$ then $g(v) = 1/a$ and if $v \in B$ then $g(v) = -1/b$. Let

$$p = \sum_{v \in V} g(v)$$

and set $f(v) = g(v) - p$. Clearly $\sum f(v) = 0$. Consider the following estimation

$$\|f\|^2 = \sum_{v \in V} (f(v))^2 \geq \sum_{v \in A \cup B} (f(v))^2 = \sum_{v \in A} \left(\frac{1}{a} - p\right)^2 + \sum_{v \in B} \left(-\frac{1}{b} - p\right)^2$$

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Note that first sum is $|A|(1/a - p)^2$ and the second sum is $|B|(-1/b - p)^2$. By expanding the squares and writing $a = |A|/|V|$ and $b = |B|/|V|$ we get
\[
\|f\|^2 \geq \frac{|V|^2}{|A|^2}|A| - 2p\frac{|V|}{|A|}|A| + p^2|A| + \frac{|V|^2}{|B|^2}|B| - 2p\frac{|V|}{|B|}|B| + p^2|B|.
\]
Note that the mid-terms cancel and we obtain
\[
\|f\|^2 \geq \frac{|V|^2}{|A|} + \frac{|V|^2}{|B|} + p^2(|A| + |B|) \geq \frac{|V|^2}{|A|} + \frac{|V|^2}{|B|} = |V|\left(\frac{1}{a} + \frac{1}{b}\right).
\]
On the other hand
\[
\langle \triangle f, f \rangle = \sum_{e \in E} (f(e_+) - f(e_-))^2 = \sum_{e \in E \setminus (E(A) \cup E(B))} (f(e_+) - f(e_-))^2
\]
since $f$ is constant on $A$ and $B$. Note that if $e_-$ is not an element of $B$ then it is not difficult to see that
\[
|f(e_+) - f(e_-)| = |g(e_+) - g(e_-)| \leq \frac{1}{\rho}\left(\frac{1}{a} + \frac{1}{b}\right).
\]
Thus
\[
\langle \triangle f, f \rangle \leq \left(|E| - |E(A)| - |E(B)|\right)\frac{1}{\rho^2}\left(\frac{1}{a} + \frac{1}{b}\right)^2.
\]
Now putting both bounds together and using the fact that $\langle \triangle f, f \rangle \geq \lambda_1(\triangle)\|f\|^2$ it is easy to see that the inequality follows.

We will apply this result to prove measure concentration of certain subsets of graphs. We first consider the growth of a set $X \subset V$ in steps. Given $r \geq 0$ we let
\[
X(r) = \{v \in V : d(v, X) \leq r\}.
\]
We use the normalized counting measure on $V$, that is, for a set $S \subset V$, $\mu(S) = |S|/|V|$. We note that if $e \in E(X, V \setminus X)$ then $e \in E(X, X(1) \setminus X)$. If in $X(1) \setminus X$ each vertex has degree at most $d$ then clearly we have
\[
|X(1) \setminus X| \geq \frac{1}{d}|E(X, X(1) \setminus X)| = \frac{1}{d}|E(X, V \setminus X)|.
\]
Now applying the previous theorem we get
\[
|X(1) \setminus X| \geq \frac{1}{d}|E(X, V \setminus X)| \geq \frac{\lambda_1(\triangle)}{d} \frac{|X||V \setminus X|}{|V|}
\]
and dividing both side by $|V|$ we get
\[
\mu(X(1) \setminus X) \geq \frac{\lambda_1(\triangle)}{d} \mu(X)(1 - \mu(X)).
\]
Since $\mu(X(1) \setminus X) = \mu(X(1)) - \mu(X)$ we get
\[
\mu(X(1)) \geq \left(1 + \frac{\lambda_1(\triangle)}{d}(1 - \mu(X))\right) \mu(X).
\]
Now assuming $\mu(X) \leq 1/2$ we obtain the following inequality

$$\mu(X(1)) \geq \left(1 + \frac{\lambda_1(\triangle)}{2d}\right) \mu(X)$$

which means that $X$ grows at a minimum rate $1 + \lambda_1(\triangle)/2d$ while it has sufficiently small measure.