High-Dimensional Measures and Geometry

Lecture Notes from April 27, 2010

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10.2 Products Of Graphs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. By product of the graphs G_1 and G_2, denoted by $G_1 \times G_2$, we mean the graph having the vertex set $V_1 \times V_2$ and the edge set defined as follows:

$$(u_1, u_2; v_1, v_2) \text{ with } u_1, v_1 \in V_1 \text{ and } u_2, v_2 \in V_2 \text{ is an edge of the product graph } G_1 \times G_2$$

$$\Updownarrow$$

$$u_1 = v_1 \text{ and } (u_2, v_2) \in E_2 \text{ or } u_2 = v_2 \text{ and } (u_1, v_1) \in E_1.$$

Example: Consider $I_1 = (\{0,1\}, (0,1))$. Then $I_1 \times I_1$ will be the graph with vertex set $V = \{(0,0), (0,1), (1,0), (1,1)\}$ and the edge set

$$E(I_1 \times I_1) = \{(0,0;0,1), (1,0;1,1), (0,0;1,0), (0,1;1,1)\}$$

which directly follows from the definition. Note that $I_1 \times I_1$ is same as I_2. In general we have

$$I_n \times I_m = I_{n+m}.$$

The Laplacian $\triangle$ of the product graph $G_1 \times G_2$ given by

$$\triangle = \triangle_1 \otimes I_{V_2} + I_{V_1} \otimes \triangle_2$$

where $\triangle_1$ and $\triangle_2$ are the Laplacian of G_1 and G_2.

Example: Consider the previous example where

$$\triangle_1 = \triangle_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\triangle = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

is the $\triangle_2$, Laplacian of I_2.

Since $\triangle_1 \otimes I_{V_2}$ and $I_{V_1} \otimes \triangle_2$ commute, the eigenvalues of $\triangle$ is the sum of eigenvalues of $\triangle_1$ and $\triangle_2$. So the smallest non-zero eigenvalue of $\triangle$ is given by

$$\lambda_1(\triangle) = \min\{\lambda_1(\triangle_1), \lambda_1(\triangle_2)\}.$$

We can iterate this tensoration and for example we can obtain the Laplacian $\triangle_n$ of I_n

$$\lambda_1(\triangle_n) = \lambda_1(\triangle_1) = \left(\frac{1}{\sqrt{2}} - (\frac{1}{\sqrt{2}})\right)^2 = 2.$$

This completes the explanation why the equality holds for I_n in preceding theorem.

We have already established that

$$\lambda_1(\triangle) \le \frac{|V|}{|V|-1}d$$

where d is the minimal degree. For I_n, $d = n$ so this bound does not contain enough information. We will construct a better estimation.

10.2.1 Theorem. *Let $G = (V, E)$ be a connected graph and $A, B \subset V$ be two disjoint subsets with distance*

$$\rho = d(A, B) = \min\{d(u, v) : \ u \in A \ v \in B\}.$$

Let $E(A)$ be the set of edges with both end points in A. Similarly define $E(B)$. Then

$$|E| - |E(A)| - |E(B)| \ge \lambda_1(\triangle)\rho^2 \frac{|A||B|}{|A| + |B|}.$$

Proof. Recall that if $f \in V^{\mathbb{R}}$ with $\sum f(v) = 0$ then $\langle \triangle f, f\rangle \ge \lambda_1(\triangle)\|f\|^2$. We will apply this result to a special function f. Let

$$a = \frac{|A|}{|V|} \text{ and } b = \frac{|B|}{|V|}$$

and consider the function g defined by

$$g(v) = \frac{1}{a} - \frac{1}{\rho}\left(\frac{1}{a} + \frac{1}{b}\right)\min\{d(v, A), \rho\}.$$

Note that if $v \in A$ then $g(v) = 1/a$ and if $v \in B$ then $g(v) = -1/b$. Let

$$p = \sum_{v \in V} g(v)$$

and set $f(v) = g(v) - p$. Clearly $\sum f(v) = 0$. Consider the following estimation

$$\|f\|^2 = \sum_{v \in V}(f(v))^2 \ge \sum_{v \in A \cup B}(f(v))^2 = \sum_{v \in A}(\frac{1}{a} - p)^2 + \sum_{v \in B}(-\frac{1}{b} - p)^2$$

Note that first sum is $|A|(1/a-p)^2$ and the second sum is $|B|(-1/b-p)^2$. By expanding the squares and writing $a=|A|/|V|$ and $b=|B|/|V|$ we get

$$\|f\|^2 \geq \frac{|V|^2}{|A|^2}|A| - 2p\frac{|V|}{|A|}|A| + p^2|A| + \frac{|V|^2}{|B|^2}|B| - 2p\frac{|V|}{|B|}|B| + p^2|B|.$$

Note that the mid-terms cancel and we obtain

$$\|f\|^2 \geq \frac{|V|^2}{|A|} + \frac{|V|^2}{|B|} + p^2(|A|+|B|) \geq \frac{|V|^2}{|A|} + \frac{|V|^2}{|B|} = |V|\left(\frac{1}{a}+\frac{1}{b}\right).$$

On the other hand

$$\langle \triangle f, f\rangle = \sum_{e\in E}\left(f(e_+)-f(e_-)\right)^2 = \sum_{e\in E\setminus(E(A)\cup E(B))}\left(f(e_+)-f(e_-)\right)^2$$

since f is constant on A and B. Note that if e_- is not an element of B then it is not difficult to see that

$$|f(e_+)-f(e_-)| = |g(e_+)-g(e_-)| \leq \frac{1}{\rho}\left(\frac{1}{a}+\frac{1}{b}\right).$$

Thus

$$\langle \triangle f, f\rangle \leq \big(|E|-|E(A)|-|E(B)|\big)\frac{1}{\rho^2}\left(\frac{1}{a}+\frac{1}{b}\right)^2.$$

Now putting both bounds together and using the fact that $\langle \triangle f, f\rangle \geq \lambda_1(\triangle)\|f\|^2$ it is easy to see that the inequality follows. $\square$

We will apply this result to prove measure concentration of certain subsets of graphs. We first consider the growth of a set $X\subset V$ in steps. Given $r\geq 0$ we let

$$X(r) = \{v\in V:\ d(v,X)\leq r\}.$$

We use the normalized counting measure on V, that is, for a set $S\subset V$, $\mu(S)=|S|/|V|$. We note that if $e\in E(X,V\setminus X)$ then $e\in E(X,X(1)\setminus X)$. If in $X(1)\setminus X$ each vertex has degree at most d then clearly we have

$$|X(1)\setminus X| \geq \frac{1}{d}|E(X,X(1)\setminus X)| = \frac{1}{d}|E(X,V\setminus X)|.$$

Now applying the previous theorem we get

$$|X(1)\setminus X| \geq \frac{1}{d}|E(X,V\setminus X)| \geq \frac{\lambda_1(\triangle)}{d}\frac{|X||V\setminus X|}{|V|}$$

and dividing both side by $|V|$ we get

$$\mu(X(1)\setminus X) \geq \frac{\lambda_1(\triangle)}{d}\mu(X)\big(1-\mu(X)\big).$$

Since $\mu(X(1)\setminus X) = \mu(X(1))-\mu(X)$ we get

$$\mu(X(1)) \geq \left(1+\frac{\lambda_1(\triangle)}{d}\big(1-\mu(X)\big)\right)\mu(X).$$

Now assuming $\mu(X) \leq 1/2$ we obtain the following inequality

$$\mu(X(1)) \geq \left(1 + \frac{\lambda_1(\triangle)}{2d}\right) \mu(X)$$

which means that X grows at a minimum rate $1 + \lambda_1(\triangle)/2d$ while it has sufficiently small measure.