High-Dimensional Measures and Geometry

Lecture Notes from April 29, 2010

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Last time we have shown that if $\mu(X) \leq 1/2$ then we have the following inequality

$$\mu(X(1)) \geq \left(1 + \frac{\lambda_1(\triangle)}{2d}\right)\mu(X)$$

which means that X grows at a minimum rate $1 + \lambda_1(\triangle)/2d$. Now by using some iteration we obtain

$$\mu(X(r+1)) \geq \left(1 + \frac{\lambda_1(\triangle)}{2d}\right)^{r+1}\mu(X)$$

as long as $\mu(X_r) \leq 1/2$. Consequently

$$\mu(X) \leq \left(1 + \frac{\lambda_1(\triangle)}{2d}\right)^{-r}\mu(X(r)) \leq \frac{1}{2}\left(1 + \frac{\lambda_1(\triangle)}{2d}\right)^{-r}.$$

Now we wish to use this result for measure concentration.

Note: If $X = V \setminus Y(r)$ for some $Y \subseteq V$ then

$$X = \{v \in V : d(v, Y) > r\}$$

We claim that $X(r) \cap Y = \emptyset$ This is because $v \in X(r)$ iff $d(v, X) \leq r$. But if $v \in Y$ then $d(v, X) \geq d(X, Y) > r$. So $v \notin Y$.

Now by going to complements, if $Y \subseteq V$ with $\mu(Y) \geq 1/2$ and $X = V \setminus Y(r)$ then $X(r) \subseteq V \setminus Y$ and $\mu(X(r)) \leq 1/2$. Thus

$$\mu(Y) \geq \frac{1}{2} \Rightarrow \mu(X) = \mu(\{v \in V : d(v, Y) > r\}) \leq \frac{1}{2}\left(1 + \frac{\lambda_1(\triangle)}{2d}\right)^{-r}.$$

Recall that for I_n, if we define

$$B = \{v \in I_n : \sum_{i=1}^{n} v_i \leq n/2\}$$

then we know

$$\mu(I_n \setminus B(\alpha n)) = \mu(\{x \in I_n : f(x) - \frac{n}{2} \geq \alpha n\}) \leq e^{-2n\alpha^2} e^{Cn\alpha^3}.$$

We compare this result with $r = \alpha n$,

$$\mu(\{v \in V : d(v, B) > r\}) \leq \frac{1}{2}\left(1 + \frac{1}{n}\right)^{-r} \leq \frac{1}{2}e^{-r/n} = \frac{1}{2}e^{-\alpha}.$$

Clearly the estimation above is better. However we can improve this estimation.

0.0.1 Theorem. *Let $G = (V, E)$ be a connected graph, $X \subseteq V$ be a set and $deg(v) \leq d$ for all v in V. Then for every $r \geq 1$, if $\mu(X(r)) \leq 1/2$, we have*

$$\mu(X(r)) \geq \left(1 + \frac{\lambda_1(\triangle)}{2d}r^2\right)\mu(X).$$

Proof. We apply the preceding edge-count estimate to $X = A$, $B = V \setminus X(r)$, then $E \setminus (E(A) \cup E(B))$ "connects" between X and $V \setminus X(r)$. Thus

$$|X(r) \setminus X| \geq \frac{1}{d}(E - E(X) - E(V \setminus X(r))) \geq \frac{\lambda_1(\triangle)r^2}{d}\frac{|X|\,|V \setminus X(r)|}{|X| + |V \setminus X(r)|}.$$

However,

$$\frac{|V \setminus X(r)|}{|X| + |V \setminus X(r)|} = \left(1 + \frac{|X|}{|V \setminus X(r)|}\right)^{-1} = \left(1 + \frac{\frac{|X|}{|V|}}{\frac{|V \setminus X(r)|}{|V|}}\right)^{-1} = \left(1 + \frac{\mu(X)}{\mu(V \setminus X(r))}\right)^{-1}$$

now considering $\mu(X) \leq 1/2$ and $\mu(V \setminus X(r)) = 1 - \mu(X(r)) \geq 1/2$ we obtain that the latter expression on the above is greater than or equal $1/2$. Hence we have

$$|X(r) \setminus X| \geq \frac{\lambda_1(\triangle)r^2}{d}\frac{|X|}{2}$$

equivalently,

$$|X(r)| \geq \frac{\lambda_1(\triangle)r^2}{d}\frac{|X|}{2} + |X| = \left(1 + \frac{\lambda_1(\triangle)}{2d}r^2\right)|X|.$$

Thus

$$\mu(X(r)) \geq \left(1 + \frac{\lambda_1(\triangle)}{2d}r^2\right)\mu(X).$$

□

As before, we conclude that if $Y \subseteq V$ with $\mu(Y) \geq 1/2$ and $X = V \setminus Y(r)$, then $\mu(X(r)) \leq 1/2$ and

$$\mu(\{v \in V : d(v, Y) > r\}) \leq \frac{1}{2}\left(1 + \frac{\lambda_1(\triangle)}{2d}r^2\right)^{-1}.$$

This estimation doesn't seem to be as good as what we had, at least for large r. But, if $\lambda_1 r/d$ is small then

$$\left(1 + \frac{\lambda_1}{d}\right)^r \cong 1 + \frac{\lambda_1 r}{2d} < 1 + \frac{\lambda_1 r^2}{2d}.$$

So this new bound is better. We iterate the bound to get good estimates when r is large.

0.0.2 Corollary. *Let $G = (V, E)$ be a connected graph such that $deg(v) \le d$ for all v in V. Let $A \subseteq V$ be a set with $\mu(A) > 1/2$. Then for any $t \in \mathbb{N}$*

$$\mu(\{v \in V : d(v, A) > t\}) \le \frac{1}{2}\left(1 + \frac{\lambda_1(\triangle)r^2}{2d}\right)^{-\lfloor t/r \rfloor}.$$

Proof. Take $B = V \setminus A(t)$. Again $B(t) \subseteq V \setminus A$ and thus $\mu(B(t)) \le 1/2$. Take $s = \lfloor t/r \rfloor$ and construct a sequence

$$X_0 = B, \;\; X_1 = X_0(r), \;\; X_2 = X_1(r), ..., X_k = X_{k-1}(r) \text{ for } k \le s.$$

Then $X_s = B(rs) \subseteq B(t)$. So $\mu(X_k) \le 1/2$ for $k = 0, 1, 2, ..., s$. Now using our refined estimate we have

$$\frac{1}{2} \ge \mu(X_s) \ge \left(1 + \frac{\lambda_1(\triangle)r^2}{2d}\right)^{s} \mu(B)$$

or

$$\mu(B) \le \frac{1}{2}\left(1 + \frac{\lambda_1(\triangle)r^2}{2d}\right)^{-s}.$$

□