Last time we have shown that if $\mu(X) \leq \frac{1}{2}$ then we have the following inequality

$$\mu(X(1)) \geq \left(1 + \frac{\lambda_1(\Delta)}{2d}\right) \mu(X)$$

which means that $X$ grows at a minimum rate $1 + \lambda_1(\Delta)/2d$. Now by using some iteration we obtain

$$\mu(X(r+1)) \geq \left(1 + \frac{\lambda_1(\Delta)}{2d}\right)^{r+1} \mu(X)$$

as long as $\mu(X_r) \leq \frac{1}{2}$. Consequently

$$\mu(X) \leq \left(1 + \frac{\lambda_1(\Delta)}{2d}\right)^{-r} \mu(X(r)) \leq \frac{1}{2} \left(1 + \frac{\lambda_1(\Delta)}{2d}\right)^{-r}. $$

Now we wish to use this result for measure concentration.

**Note:** If $X = V \setminus Y(r)$ for some $Y \subseteq V$ then

$$X = \{v \in V : d(v,Y) > r\}$$

We claim that $X(r) \cap Y = \emptyset$. This is because $v \in X(r)$ iff $d(v,X) \leq r$. But if $v \in Y$ then $d(v,X) \geq d(X,Y) > r$. So $v \notin Y$.

Now by going to complements, if $Y \subseteq V$ with $\mu(Y) \geq \frac{1}{2}$ and $X = V \setminus Y(r)$ then $X(r) \subseteq V \setminus Y$ and $\mu(X(r)) \leq \frac{1}{2}$. Thus

$$\mu(Y) \geq \frac{1}{2} \Rightarrow \mu(X) = \mu(\{v \in V : d(v,Y) > r\}) \leq \frac{1}{2} \left(1 + \frac{\lambda_1(\Delta)}{2d}\right)^{-r}. $$

Recall that for $I_n$, if we define

$$B = \{v \in I_n : \sum_{i=1}^n v_i \leq n/2\}$$

then we know

$$\mu(I_n \setminus B(\alpha n)) = \mu(\{x \in I_n : f(x) - \frac{n}{2} \geq \alpha n\}) \leq e^{-2\alpha^2} e^{Cn\alpha^3}. $$
We compare this result with \( r = \alpha n \),
\[
\mu(\{v \in V : d(v, B) > r\}) \leq \frac{1}{2} \left(1 + \frac{1}{n}\right)^{-r} \leq \frac{1}{2} e^{-r/n} = \frac{1}{2} e^{-\alpha}.
\]

Clearly the estimation above is better. However we can improve this estimation.

**0.0.1 Theorem.** Let \( G = (V, E) \) be a connected graph, \( X \subseteq V \) be a set and \( \deg(v) \leq d \) for all \( v \) in \( V \). Then for every \( r \geq 1 \), if \( \mu(X(r)) \leq 1/2 \), we have
\[
\mu(X(r)) \geq \left(1 + \frac{\lambda_1(\triangle)r^2}{2d}\right) \mu(X).
\]

**Proof.** We apply the preceding edge-count estimate to \( X = A, B = V \setminus X(r) \), then \( E \setminus (E(A) \cup E(B)) \) "connects" between \( X \) and \( V \setminus X(r) \). Thus
\[
|X(r) \setminus X| \geq \frac{1}{d}(E - E(X) - E(V \setminus X(r))) \geq \frac{\lambda_1(\triangle)r^2}{d} \frac{|X| |V \setminus X(r)|}{|X| + |V \setminus X(r)|}.
\]

However,
\[
\frac{|V \setminus X(r)|}{|X| + |V \setminus X(r)|} = \left(1 + \frac{|X|}{|V \setminus X(r)|}\right)^{-1} = \left(1 + \frac{|X|}{\frac{|V \setminus X(r)|}{|V|}}\right)^{-1} = \left(1 + \frac{\mu(X)}{\mu(V \setminus X(r))}\right)^{-1}
\]

now considering \( \mu(X) \leq 1/2 \) and \( \mu(V \setminus X(r)) = 1 - \mu(X(r)) \geq 1/2 \) we obtain that the latter expression on the above is greater than or equal 1/2. Hence we have
\[
|X(r) \setminus X| \geq \frac{\lambda_1(\triangle)r^2}{d} \frac{|X|}{2}
\]
equivalently,
\[
|X(r)| \geq \frac{\lambda_1(\triangle)r^2}{d} \frac{|X|}{2} + |X| = \left(1 + \frac{\lambda_1(\triangle)r^2}{2d}\right)|X|.
\]

Thus
\[
\mu(X(r)) \geq \left(1 + \frac{\lambda_1(\triangle)r^2}{2d}\right) \mu(X).
\]

As before, we conclude that if \( Y \subseteq V \) with \( \mu(Y) \geq 1/2 \) and \( X = V \setminus Y(r) \), then \( \mu(X(r)) \leq 1/2 \) and
\[
\mu(\{v \in V : d(v, Y) > r\}) \leq \frac{1}{2} \left(1 + \frac{\lambda_1(\triangle)r^2}{2d}\right)^{-1}.
\]

This estimation doesn’t seem to be as good as what we had, at least for large \( r \). But, if \( \lambda_1r/d \) is small then
\[
\left(1 + \frac{\lambda_1}{d}\right)^r \approx 1 + \frac{\lambda_1r}{d} < 1 + \frac{\lambda_1r^2}{2d}.
\]

So this new bound is better. We iterate the bound to get good estimates when \( r \) is large.
0.0.2 Corollary. Let $G = (V, E)$ be a connected graph such that $\text{deg}(v) \leq d$ for all $v$ in $V$. Let $A \subseteq V$ be a set with $\mu(A) > 1/2$. Then for any $t \in \mathbb{N}$

$$\mu(\{v \in V : d(v, A) > t\}) \leq \frac{1}{2} \left(1 + \frac{\lambda_1(\Delta)r^2}{2d}\right)^{-\lfloor t/r \rfloor}.$$ 

Proof. Take $B = V \setminus A(t)$. Again $B(t) \subseteq V \setminus A$ and thus $\mu(B(t)) \leq 1/2$. Take $s = \lfloor t/r \rfloor$ and construct a sequence

$$X_0 = B, \; X_1 = X_0(r), \; X_2 = X_1(r), \ldots, X_k = X_{k-1}(r) \text{ for } k \leq s.$$ 

Then $X_s = B(rs) \subseteq B(t)$. So $\mu(X_k) \leq 1/2$ for $k = 0, 1, 2, \ldots, s$. Now using our refined estimate we have

$$\frac{1}{2} \geq \mu(X_s) \geq \left(1 + \frac{\lambda_1(\Delta)r^2}{2d}\right)^s \mu(B).$$

or

$$\mu(B) \leq \frac{1}{2} \left(1 + \frac{\lambda_1(\Delta)r^2}{2d}\right)^{-s}.$$ 

\qed