Practice Exam #2
Math 1450

Fall 2014
Section ____________

Name: ________________________________ **Student ID:** ________________________

1. (a) State a property of a function f defined on an interval $[a, b]$ which ensures that f is integrable on $[a, b]$. Choose a property which shows that many kinds of functions are integrable.

   (b) Assuming f is integrable on $[a, b]$, state an expression for the integral of f from a to b as a limit of sums.

2. Suppose f has a continuous derivative on $[0, 5]$, $f(0) = 2$ and $1 \leq f^{'}(x) \leq 2$ for all x in $[0, 5]$. With the help of known facts from class, show that

$$f(5) \leq 12\,.$$

3. The velocity of a particle at time t is $v(t) = 3t - 5$.

   (a) Find the displacement of the particle from $t = 0$ to $t = 3$.

   (b) Find the total distance traveled by the particle from $t = 0$ to $t = 3$.

4. Evaluate the following definite or indefinite integrals:

(a)

$$\int_0^{\pi/2} \sin(2x)\cos(x)dx$$

(b)

$$\int (1 + 2\tan(t))^2 \sec^2(t)dt$$

(c)

$$\int_0^4 |x-2|dx$$

(d)

$$\int \frac{x^2}{(x+2)^3}dx$$

(e)

$$\int_1^3 x\log(x)dx$$

(f)

$$\int \tan^3(x)\sec^3(x)dx$$

5. Use a trigonometric substitution to compute $\int x^3\sqrt{x^2-1}dx$.

6. Use integration by parts and the identity $\sin^2 x + \cos^2 x = 1$ to relate the indefinite integral $I_n := \int \sin^n x dx$ to I_{n-2}, where $n \geq 2$ is an even integer.

7. Decompose $\dfrac{5x^2 + 6x + 4}{x^3 + x^2 - 2}$ into partial fractions. As a first step, try to guess a zero of the polynomial in the denominator.

8. Show, by considering a Riemann sum or otherwise, that for each positive integer n,

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} \leq 1$$

9. By comparing $\frac{x^2}{x^5+x^2+1}$ to a simpler function show that

$$\int_0^1 \frac{x^2}{x^5+x^2+1}\,dx \le \frac{1}{4}$$

10. Show that if f is a continuous, even function and F an antiderivative of f which has the value $F(0) = 0$, then F is an odd function. Hint: FTC and substitution.

11. Does $\int_0^1 \ln x dx$ exist? If so, how is this integral defined?

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1. (a) State a property of a function f defined on an interval $[a, b]$ which ensures that f is integrable on $[a, b]$. Choose a property which shows that many kinds of functions are integrable.

If f is continuous on $[a,b]$ except for finitely many jump discontinuities, then f is integrable on $[a,b]$.

(b) Assuming f is integrable on $[a, b]$, state an expression for the integral of f from a to b as a limit of sums.

If f is integrable, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\,\Delta x$$

with $\Delta x = \frac{b-a}{n}$,

$x_j = a + j\Delta x$

2. Suppose f has a continuous derivative on $[0,5]$, $f(0)=2$ and $1 \le f'(x) \le 2$ for all x in $[0,5]$. With the help of known facts from class, show that

$$f(5) \le 12\,.$$

By differentiability of f on $[0,5]$ and continuity of f' on $[0,5]$,

$$f(5) - f(2) = \int_0^5 f'(x)\,dx\,.$$

Comparing f' gives by $f'(x) \le 2$

$$f(5) - f(2) \le \int_0^5 (2)\,dx$$

$$= 2x \Big|_0^5 = 10$$

so

$$f(5) \le f(2) + 10 = 12\,.$$

3. The velocity of a particle at time t is $v(t) = 3t - 5$.

(a) Find the displacement of the particle from $t = 0$ to $t = 3$.

$$s(3) - s(0) = \int_0^3 \underbrace{(3t-5)}_{v(t)}\,dt$$

$$= \left[\frac{3}{2}t^2 - 5t\right]_0^3 = \frac{27}{2} - 15 = -\frac{3}{2}$$

(b) Find the total distance traveled by the particle from $t = 0$ to $t = 3$.

$$\text{distance} = \int_0^3 |v(t)|\,dt$$

$$= \int_0^{5/3} -(3t-5)\,dt + \int_{5/3}^3 (3t-5)\,dt$$

$$= -\left[\frac{3}{2}t^2 - 5t\right]_0^{5/3} + \int_0^{4/3} 3u\,du$$

$$= -\frac{3}{2}\frac{25}{9} + \frac{25}{3} + 3\,\frac{1}{2}\left(\frac{4}{3}\right)^2$$

$$= \frac{25}{6} + \frac{8}{3} = \frac{41}{6}$$

4. Evaluate the following definite or indefinite integrals:

(a)

$$\int_0^{\pi/2} \underbrace{\sin(2x)}_{2\sin x\cos x} \cos(x)dx$$

$u = \cos x$

$-du = +\sin x\,dx$

$$= 2\int_0^{\pi/2} \cos^2 x \sin x\,dx$$

$$= 2\int_1^0 u^2(-du) = 2\int_0^1 u^2\,du$$

$$= \frac{2}{3}.$$

(b)

$$\int (1 + 2\tan(t))^2 \sec^2(t)dt$$

$u = \tan t$

$du = \sec^2 t\,dt$

$$= \int (1+2u)^2\,du$$

$$= \int (1 + 4u + 4u^2)\,du$$

$$= \left[u + 2u^2 + \frac{4}{3}u^3\right] + C$$

$$= \tan t + 2\tan^2 t + \frac{4}{3}\tan^3 t + C,$$

(c)

$$\int_0^4 |x-2|dx$$

$$= \int_0^2 \underbrace{|x-2|}_{-(x-2)} dx + \int_2^4 \underbrace{|x-2|}_{x-2} dx$$

$$= -\left[\frac{x^2}{2} - 2x\right]_0^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$= -\frac{4}{2} + 4 + \frac{16}{2} - 8 - \frac{4}{2} + 4 = 2 + 2 = 4$$

(d)

$$\int \frac{x^2}{(x+2)^3} dx \qquad u = x+2$$

$$= \int \frac{(u-2)^2}{u^3} du = \int \frac{u^2 - 4u + 4}{u^3} du$$

$$= \int \left[u^{-1} - 4u^{-2} + 4u^{-3}\right] du$$

$$= \ln|u| + 4u^{-1} - 2u^{-2} + C$$

$$= \ln|x+2| + \frac{4}{x+2} - 2\frac{1}{(x+2)^2} + C.$$

(e)

$$\int_1^3 x\log(x)dx \qquad u = \ln x \quad v = x^2/2$$
$$u' = \frac{1}{x} \quad v' = x$$

$$= \frac{x^2}{2}\ln x \Big|_1^3 - \int_1^3 \frac{x^{\cancel{2}}}{2}\frac{1}{\cancel{x}} dx$$

$$= \frac{9}{2}\ln 3 - \left[\frac{x^2}{4}\right]_1^3 = \frac{9}{2}\ln 3 - \frac{9}{4} + \frac{1}{4}$$

$$= \frac{9}{2}\ln 3 - 2$$

(f)

$$\int \tan^3(x)\sec^3(x)dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$= \int \tan^2 x \sec^2 x \quad \sec x \tan x \, dx$

$= \int (\sec^2 x - 1) \sec^2 x \quad \sec x \tan x \, dx$

$= \int (u^2 - 1) u^2 \, du$

$= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$

$= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$

5. Use a trigonometric substitution to compute $\int x^3\sqrt{x^2-1}dx$.

$$x = \sec\theta$$
$$dx = \sec\theta\tan\theta\, d\theta$$

$$\int x^3\sqrt{x^2-1}\,dx$$

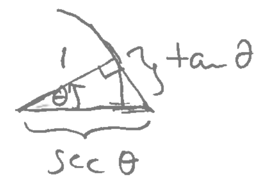
$$= \int \sec^3\theta \underbrace{\sqrt{\sec^2\theta - 1}}_{\tan\theta} \sec\theta\tan\theta\, d\theta$$

$$= \int \sec^4\theta \tan^2\theta \, d\theta = \int (1+\tan^2\theta)\tan^2\theta\sec^2\theta\, d\theta$$

$u = \tan\theta$

$$= \int (1+u^2)u^2\, du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + C$$



$$= \frac{1}{3}(x^2-1)^{3/2} + \frac{1}{5}(x^2-1)^{5/2} + C$$

6. Use integration by parts and the identity $\sin^2 x + \cos^2 x = 1$ to relate the indefinite integral $I_n := \int \sin^n x dx$ to I_{n-2}, where $n \geq 2$ is an even integer.

$$u = \sin^{n-1}x$$
$$u' = (n-1)\sin^{n-2}x\cos x$$
$$v = -\cos x$$
$$v' = \sin x$$

$$I_n = \int \sin^{n-1}x \sin x\, dx$$

$$= -\sin^{n-1}x\cos x + \int (n-1)\sin^{n-2}x\cos^2 x\, dx$$

$$= -\sin^{n-1}x\cos x + \underbrace{(n-1)\int \sin^{n-2}x(1-\sin^2 x)\,dx}_{(n-1)I_{n-2} - (n-1)I_n}$$

$$\cancel{n} I_n = \frac{1}{n}\left(-\sin^{n-1}x\cos x + (n-1)I_{n-2}\right).$$

7. Decompose $\dfrac{5x^2+6x+4}{x^3+x^2-2}$ into partial fractions. As a first step, try to guess a zero of the polynomial in the denominator. $x = 1$ is root

$$
\begin{array}{r l}
 & x^2 + 2x + 2 \\ \hline
x-1 \,) & x^3 + x^2 + 0x - 2 \\
 & x^3 - x^2 \\ \hline
 & 2x^2 \cancel{+2x} \\
 & 2x^2 - 2x \\ \hline
 & 2x - 2 \\
 & 2x - 2 \\ \hline
 & 0
\end{array}
$$

$(x+1)^2 + 1 \geq 1$
not further red.

So $$\frac{5x^2+6x+4}{x^3+x^2-2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2}$$

$$5x^2+6x+4 = (x^2+2x+2)A + (x-1)(Bx+C)$$

putting $x=1$ gives

$$15 = 5A \quad \Rightarrow \quad A = 3$$

Next,

$$5x^2+6x+4 - 3(x^2+2x+2)$$

$$= 2x^2 + 0x - 2 = Bx^2 + (-B+C)x - C$$

$$\Rightarrow \quad B = 2, \quad C = +2.$$

We conclude

$$\frac{5x^2+6x+4}{x^3+x^2-2} = \frac{3}{x-1} + \frac{2x+2}{x^2+2x+2}$$

9. By comparing $\frac{x^2}{x^5+x^2+1}$ to a simpler function show that

$$\int_0^1 \frac{x^2}{x^5+x^2+1}\,dx \le \frac{1}{4}$$

If $0 \le x \le 1$, then $x^5 \ge 0$, so

$$\frac{x^2}{x^5+x^2+1} \le \frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}.$$

$$\int_0^1 \frac{x^2}{x^5+x^2+1}\,dx \le \int_0^1 \left(1 - \frac{1}{x^2+1}\right)dx$$

$$= 1 - \left[\tan^{-1}(x)\right]_0^1$$

$$= 1 - \underbrace{\tan^{-1}(1)}_{\frac{\pi}{4}}.$$

Since $\pi \ge 3$, $1 - \frac{\pi}{4} \le 1 - \frac{3}{4} = \frac{1}{4}$.

8. Show, by considering a Riemann sum or otherwise, that for each positive integer n,

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} \leq 1$$

LHS is Riemann sum with $\Delta x = 1$ on $[a,b]$, $a = 1$, $b = n$ with

$$x_j = 1 + j\Delta x, \quad f(x) = \frac{1}{x^2}.$$

Since $\frac{1}{x^2}$ is decreasing and x_j right endpoint of interval $[1+(j-1)\Delta x, 1+j\Delta x]$,

$$(\Delta x) \sum_{j=1}^{n-1} \frac{1}{(1+j\Delta x)^2} = \sum_{j=1}^{n-1} \frac{1}{(1+j)^2} \leq \int_1^n \frac{1}{x^2}\,dx$$

We see that

$$\int_1^n \frac{1}{x^2}\,dx \leq \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left[1 - \frac{1}{b}\right]$$

$$= 1$$

Thus, $\displaystyle\sum_{j=1}^{n-1} \frac{1}{(1+j)^2} \leq 1.$

10. Show that if f is a continuous, even function and F an antiderivative of f which has the value $F(0) = 0$, then F is an odd function. Hint: FTC and substitution.

$$F(x) = F(x) - F(0)$$

$$\overset{\text{FTC}}{=} \int_0^x f(t)\,dt$$

$\downarrow$ f even

$$= \int_0^x f(-t)\,dt$$

$u = -t$

$$= -\int_0^{-x} f(u)\,du$$

$$= -\left(F(-x) - F(0)\right)$$

$$= -F(-x)$$

so F is odd.

11. Does $\int_0^1 \ln x dx$ exist? If so, how is this integral defined?

$$\int_0^1 \ln x \, dx = \lim_{a \to 0^+} \int_a^1 \ln x \, dx$$

$u = \ln x \quad v = x$

$u' = \frac{1}{x} \quad v' = 1$

$$= \lim_{a \to 0^+} \left[x \ln x \Big|_a^1 - \int_a^1 \frac{x}{x} \, dx \right]$$

$$= \lim_{a \to 0^+} \left[- a \ln a - (1-a) \right]$$

$$\overset{l'H}{=} \lim_{a \to 0^+} \left[- \frac{\ln a}{\frac{1}{a}} - (1-a) \right]$$

$$= \lim_{a \to 0^+} \left[\underbrace{\frac{\frac{1}{a}}{\frac{1}{a^2}}}_{a} - (1-a) \right]$$

$$= -1$$

so integral is convergent,