Math 3364 Practice Final

December, 2010
Two hours and twenty minutes
University of Houston

Instructions:

1. Put your name, signature and I.D. No. in the blanks above.

2. Answer the questions in the spaces provided, using the backs of pages or the blank page at the end for overflow or rough work.

3. Your grade will be influenced by how clearly you present your solutions. **Justify your solutions carefully** by referring to definitions and results from class where appropriate.

4. No calculators or cell phones permitted.

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(15 points) 1. (a) Find the argument of the following number and write it in polar form:

\[
\arg[(1 - i)(-\sqrt{3} + i)] = \frac{5\pi}{6} - \frac{\pi}{4} = \frac{10 - 3\pi}{12} = \frac{7\pi}{12}
\]

\[
= 2\sqrt{2} e^{7i\pi/12}
\]

\[
(1-i)(-\sqrt{3}+i) = 2\sqrt{2} e^{7i\pi/12}
\]

(b) True or false: Every power series at \( z_0 \) with radius of convergence \( R > 0 \) converges uniformly to an analytic function \( f(z) \) on the open disk \( \{ z \in \mathbb{C} : |z - z_0| < R \} \).

\[
\text{circle: TRUE or FALSE.}
\]

\[
\text{convergence \hspace{1cm} yes}
\]

\[
\text{uniform \hspace{1cm} no}
\]

(c) The residue of \( ze^{z^3} \) at \( z = 0 \) is:

\[
f(z) = z e^{z^3} \rightarrow \text{analytic} \Rightarrow \text{Res}(0) = 0.
\]
(10 points) 2. What equation(s) do $x$ and $y$ have to satisfy if $z = x + iy$ satisfies $z^2 + \bar{z}^2 = 2$?

\[
(x + iy)^2 + (x - iy)^2 = 2
\]

\[
x^2 + 2ixy - y^2 + x^2 - 2ixy - y^2 = 2
\]

\[
\Rightarrow 2x^2 - 2y^2 = 2
\]

\[
x^2 - y^2 = 1
\]
(15 points) 3. Take \( u(x, y) = xy - x + y + x^2 - ky^2 \), where the constant \( k \) is a real number.

(a) If \( u \) is harmonic in the entire plane, what condition holds for the constant \( k \)?

\[
\frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = -2k
\]

\[
\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2k = 0 \Rightarrow k = 1.
\]

(b) Assuming the constant \( k \) has been chosen so that \( u \) is harmonic, what is its harmonic conjugate \( v \) (up to a constant)?

\[
\frac{\partial u}{\partial x} = y - 1 + 2x = \frac{\partial v}{\partial y}
\]

\[
\Rightarrow v = \frac{1}{2} y^2 - y + 2xy + g(x)
\]

\[
\frac{\partial v}{\partial x} = 2y + g'(x) = -\frac{\partial u}{\partial y} = -(x + 1 - 2y)
\]

\[
\Rightarrow g'(x) = -x - 1
\]

\[
\Rightarrow g(x) = -\frac{1}{2} x^2 - x + C
\]

\[
\Rightarrow v(x, y) = \frac{1}{2} y^2 - y + 2xy - \frac{1}{2} x^2 - x + C
\]
(10 points) 4. Consider \( f(z) = e^{|z|^2} \). Find all the points where \( f \) is analytic.

\[
\text{CR diff. eq. is: }
\]

\[
f(z) = \Re \left[ f(z) \right] = u(x, y), \quad z = x + iy
\]

\[
\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{3}{\partial x} e^{x^2+y^2} = 2x e^{x^2+y^2}
\]

\[
= \frac{\partial v}{\partial y} = 0 \quad \text{(b/c } \quad v = 0)\]

\[
\frac{\partial u}{\partial y} = 2y e^{x^2+y^2} = -\frac{\partial v}{\partial x} = 0
\]

\[
\Rightarrow \quad 2x e^{x^2+y^2} = 0 \quad \text{and} \quad 2y e^{x^2+y^2} = 0
\]

by continuity of partial derivs.,

\( f \) has derivative at \( z = 0 \) only.

This is an isolated point, not open set, so \( f \) is analytic nowhere.
(c) Assuming the constant $k$ has been chosen so that $u$ is harmonic, find the function $f(z)$ such that $f(x + iy) = u(x, y) + iv(x, y)$ and express it in terms of $z$.

\[
\begin{align*}
    u + iv &= x^2 - y^2 - x + y + xy \\
    &= 2 + x^2 - y^2 - x - y - \frac{1}{2} i (x^2 - y^2) \\
    &= z^2 - 1 i z - \frac{1}{2} z^2 \\
    &= (1 - \frac{i}{2}) z^2 - (1 + i) z.
\end{align*}
\]
(10 points) 5. Use Cauchy’s formula to prove that if \( f \) is entire, then for any \( R > 0 \),

\[
f(0) = \frac{1}{2\pi i} \oint_{C_R} f(z) \frac{dz}{z}.
\]

Cauchy

\[
f(0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z} \, dz
\]

parametrize \( C_R \): \( \phi(t) = Re^{it} \), \( 0 \leq t \leq 2\pi \)

\[
f(0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(Re^{it})}{Re^{it}} \, Re^{it} \, dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} f(Re^{it}) \, dt.
\]
(10 points) 6. Calculate the Laurent series for the function \( f(z) = \frac{1}{2z^2} + \frac{1}{z-2} \) and \( z \) in the annulus \( \sqrt{2} < |z| < 2 \).

\[
\frac{1}{2+z^2} \quad \text{singular at } z = \pm i \sqrt{2}
\]

Factor out \( \frac{1}{z^2} \), then if \( \sqrt{2} < |z| < 1 \):

\[
\frac{1}{2+z^2} = \frac{1}{z^2} \frac{1}{1 + \frac{2}{z^2}} = \frac{1}{z^2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{2^j z^{-2j}}
\]

\[
= \frac{1}{z^2} \sum_{n=0}^{\infty} (-1)^n \left( \frac{2}{z^2} \right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{z^{2n+2}}
\]

and

\[
\frac{1}{z-2} = \frac{1}{z} \frac{1}{1 - \frac{2}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \left( \frac{2}{z} \right)^n
\]

Laurent series is

\[
\sum_{n=-\infty}^{\infty} c_n z^n
\]

with

\[
c_n = \begin{cases} 
\frac{1}{2^{n+1}}, & n \geq 0 \\
0, & n < 0, \text{ n odd}
\end{cases}
\]

\[
(-1)^{\frac{n}{2}+1} 2^{-\frac{n+1}{2}}, & n < 0, \text{ n even}
\]
7. Find all the solutions of the equation \( \sin(z) = \cos(z) \) and express them in terms of the (multi-valued) logarithm of suitable complex numbers.

\[
\frac{1}{2i} \left( e^{iz} - e^{-iz} \right) = \frac{1}{z} \left( e^{iz} + e^{-iz} \right)
\]

\[
\left( \frac{1}{2i} - \frac{1}{z} \right) e^{iz} = \left( \frac{1}{2i} + \frac{1}{z} \right) e^{-iz}
\]

\[
e^{2iz} = \frac{-i + 1}{-i - 1}
\]

\[
\Rightarrow z = \frac{1}{2i} \log \left( \frac{1 + i}{1 - i} \right)
\]
8. Power and Taylor series

(a) (5 points) Find the radius of convergence of the power series

\[ \sum_{j=1}^{\infty} \frac{1}{j^2} z^j \]

Quotient test \[ \left| \frac{\frac{1}{(j+1)^2}}{\frac{1}{j^2}} \right| = \frac{j^2}{(j+1)^2} \]

\[ \Rightarrow |z| < 1 \]

\[ \Rightarrow R = 1 \]

(b) (10 points) Given that

\[ \frac{1}{1-w} = \sum_{j=0}^{\infty} w^j \]

for \(|w| < 1\), find the Maclaurin series for

\[ f(z) = \log(1 - z^2) \]

and state its radius of convergence. Refer to results discussed in class to explain your answer.

Integrate term by term

\[ -\log(1 - w) = \sum_{j=0}^{\infty} \frac{1}{j+1} w^{j+1} \]

\[ \log(1 - w) = -\sum_{j=1}^{\infty} \frac{1}{j} w^j, \quad |w| < 1 \]

Input \( z^2 = w \)

\[ \log(1 - z^2) = \sum_{j=1}^{\infty} \left(-\frac{1}{j}\right) z^{2j} \]

Then radius \( R = 1 \), b/c \( |z^2| = |w| < 1 \).
(15 points) 9. Let \( \Gamma \) be the unit circle, traversed counterclockwise. Evaluate the integral

\[
I = \int_{\Gamma} \frac{e^z}{z(3z-1)} \, dz.
\]

\[
f(z) = \frac{e^z}{z(3z-1)} \text{ is not analytic at}
\]

\[
z = 0, \quad z = \frac{1}{3}
\]

at \( z = 0 \), simple pole
at \( z = \frac{1}{3} \), pole of order 2

\[
\Rightarrow \quad I = 2\pi i \left[ \text{Res}(0) + \text{Res} \left( \frac{1}{3} \right) \right]
\]

\[
\text{Res}(0) = \frac{e^{0}}{1} = 1
\]

\[
\text{Res} \left( \frac{1}{3} \right) = \lim_{z \to \frac{1}{3}} \frac{1}{z} \left. \frac{d}{dz} \left[ \frac{(z-\frac{1}{3})^2 e^z}{z(3z-1)} \right] \right|_{z = \frac{1}{3}}
\]

\[
= \lim_{z \to \frac{1}{3}} \frac{1}{9} \left[ \frac{e^z}{z} - \frac{e^z}{z^2} \right]
\]

\[
= \frac{1}{9} \left[ 3e^{\frac{1}{3}} - 9e^{\frac{1}{3}} \right]
\]

\[
= \left( \frac{1}{3} - 1 \right) e^{\frac{1}{3}}
\]

\[
\Rightarrow \quad I = 2\pi i \left( 1 + \frac{1}{3} e^{\frac{1}{3}} - e^{\frac{1}{3}} \right)
\]
(15 points) 10. Evaluate the integral

\[ I = \int_{-\infty}^{\infty} \frac{1}{x^2 - 2x + 5} \, dx. \]

Explain briefly how to relate \( I \) to the value of a contour integral, and then use residues to compute it.

Consider \( T_R \),

\[ \lim_{R \to \infty} \int_{P_R} \frac{1}{z^2 - 2z + 5} \, dz = I \]

because

\[ \left| \int_{C_R} \frac{1}{z^2 - 2z + 5} \, dz \right| \leq \int_{0}^{\pi} \frac{R}{R^2 - 2R - 5} \, d\theta \]

So

\[ I = 2\pi i \sum_{j=1}^{n} \text{Res} \left( \frac{1}{z^2 - 2z + 5} \right) \]

\[ z^2 - 2z + 5 = 0 \quad \Rightarrow \quad z_{1,2} = 1 \pm \sqrt{1 - 5} = 1 \pm 2i \]

only \( 1 + 2i \) linde contour, so

\[ \text{Res} \left( 1 + 2i \right) = \frac{1}{1 + 2i - (1 - 2i)} = \frac{1}{4i} \]

\( \Rightarrow \)

\[ I = 2\pi i \frac{1}{4i} = \frac{\pi}{2} \]