Math 3364 Practice Midterm 1

Some time in September, 2010
One Hour and twenty minutes
University of Houston

Instructions:

1. Put your name, signature and I.D. No. in the blanks above.

2. Answer the questions in the spaces provided, using the backs of pages or the blank page at the end for overflow or rough work.

3. Your grade will be influenced by how clearly you present your solutions. Justify your solutions carefully by referring to definitions and results from class where appropriate.

4. No calculators permitted.

<table>
<thead>
<tr>
<th>Question</th>
<th>Value</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>110</td>
<td></td>
</tr>
</tbody>
</table>
(15 points) 1. Express the following numbers in the form $x + iy$:

(a) $\frac{1 + 2i}{3 - 4i} + \frac{2 - i}{5i} =$

(b) $(1 + i)^{10} =$

(c) $\frac{i}{-2 - 2i} =$
(10 points) 2. Show that if two complex numbers $z_1, z_2$ satisfy $\text{Im}(z_1 z_2) = 0$ and neither equals zero, then there is a real number $\lambda$ such that $z_2 = \lambda z_1$. Hint: use the polar form.
(10 points) 3. Recall that the principal sixth root of unity is $\omega_6^1 = e^{\pi i/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

(a) (6 points) Compute all sixth roots of 64, that is, all $z$’s which give $z^6 = 64$, in the form $z = x + iy$.

(b) (4 points) Sketch the location of all these roots in the complex plane.
4. Consider the quadratic function $u(x, y) = x^2 + axy + by^2$, where the two constants $a$ and $b$ are real.

(a) If $u$ is harmonic in the entire plane, what condition(s) hold(s) for the constants?

(b) Assuming the constants $a$ and $b$ have been chosen so that $u$ is harmonic, what is its harmonic conjugate $v$ (up to a constant)?

(c) Assuming the constants $a$ and $b$ have been chosen so that $u$ is harmonic, find the function $f(z)$ such that $f(x + iy) = u(x, y) + iv(x, y)$ and express it in terms of $z$. 

(15 points) 5. Show that if \( u \) is harmonic in the entire plane and \( v \) is its harmonic conjugate, then \( h(x + iy) = u(x, y)v(x, y) \) also defines a harmonic function. Hint: use the Cauchy-Riemann differential equations.
(10 points) 6. Discuss the analyticity of $f(x + iy) = x^2 + y^2 + y - 5 + ix$
7. Consider the function $f(z) = z + \frac{1}{z}$.

(a) Find all the points in the complex plane where $\text{Re}(f(z)) = 0$.

(b) Find all the points in the complex plane where $\text{Im}(f(z)) = 0$. Describe this set geometrically.

(c) Considering these level sets from (a) and (b), what do you know about their intersections?
(10 points) 8. Take $f(z) = e^z$ defined on the domain $0 \leq \text{Re}(z) \leq \log(2)$ and $21\pi \leq \text{Im}(z) \leq 23\pi$. What is the range of $f$ for this choice of domain?
(10 points) 9. Derive an expression for $\cos(\theta_1 + \theta_2)$ in terms of $\cos(\theta_1)$, $\cos(\theta_2)$, $\sin(\theta_1)$ and $\sin(\theta_2)$ with the help of complex exponentials.