Math4310/6317, Fall 2011
Problem Set 5, due Thursday, September 29

Solutions

Problem 1. Suppose we study the number of times a student sits in a classroom with a TB infected, coughing neighbor until the student contracts the disease. Assume that each classroom encounter is an independent Bernoulli trial with probability p that the student becomes infected. This leads to the so-called geometric distribution $P(\text{Person is infected on encounter } x) = p(1-p)^{x-1}$ for $x = 1, 2, \ldots$.

a. The likelihood function for an observed x is

$$\mathcal{L}(p, x) = p(1-p)^{x-1}$$

so minimizing the log-likelihood

$$\ell(p, x) = \log p + (x-1)\log(1-p)$$

gives the (only) critical point

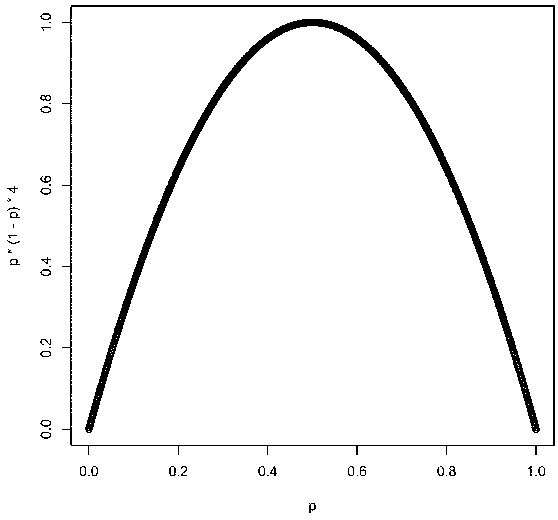
$$p = \frac{1}{x}\,.$$

At $p = 0$ or $p = 1$, the likelihood is zero, but at $p = 1/x$ it is positive, so this is the MLE.

b. Given $x = 2$. We use

```
p<-seq(0,1,0.001)
plot(p,p*(1-p)*4)
```

to plot the (normalized) likelihood. It seems that many p values are comparable (likelihood ratios, say, below 8) with the MLE.



c. We compute $P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3) = p + p(1-p) + p(1-p)^2 = 0.0297$, so this would be a rare event.

d. Assuming all individuals are infected in an iid manner, then the likelihood function is the product

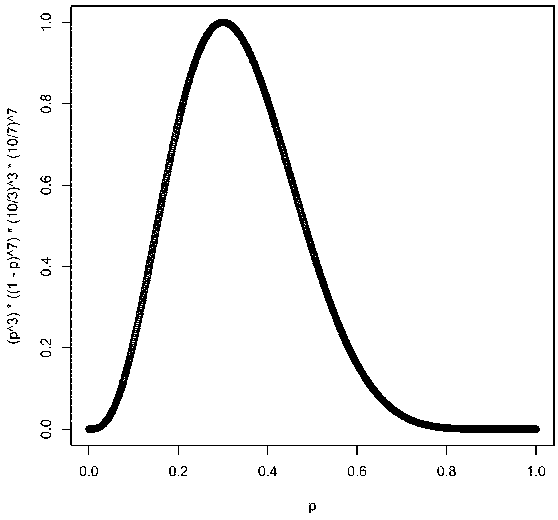
$$\mathcal{L}(p, x_1, x_2, \ldots x_n) = p(1-p)^{x_1-1} p(1-p)^{x_2-1} \cdots p(1-p)^{x_n-1} = p^n (1-p)^{\sum_i x_i - n}$$

and the log likelihood

$$\ell(p, x_1, x_2, \ldots x_n) = n \log p + (\sum_{i=1}^{n} x_i - n) \log(1-p)$$

Again searching for a critical point gives $p = \frac{n}{\sum x_i} = \frac{1}{\overline{X_n}}$, with a positive likelihood, while at $p = 0$ or $p = 1$ the likelihood vanishes. Therefore, $1/\overline{X_n}$ is the MLE.

e. The likelihood function is now proportional to $p^3(1-p)^7$. The plot shows that more values for p can now be excluded by taking the likelihood ratio with the MLE.



Problem 2. Assuming that the results for each egg are independent and identically distributed:

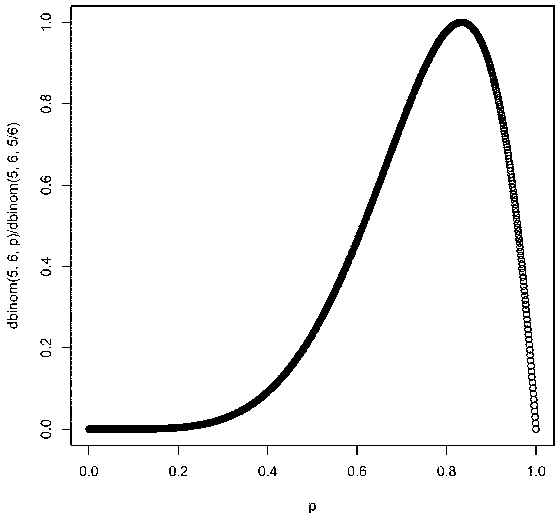
a. By the binomial distribution,

$$P(X \geq 5) = P(X = 5) + P(X = 6) = 6 * (0.1)^5 * (0.9) + (0.1)^6 = 0.000055$$

Observing this event is extremely rare under normal conditions.

b. We have the MLE estimate given by the proportion $p = x/n = 5/6$.

c. The plot gives



so comparing the likelihood at $p = 0.1$ with that at $p = 5/6$ gives the ratio

$$\frac{(5/6)^5(1/6)}{(0.1)^5(0.9)} = 7442\,.$$

This means there is 7442 times the evidence supporting the hypothesis $p = 5/6$ compared to the hypothesis $p = 0.1$.

Problem 3. Suppose that IQs in a particular population are normally distributed with a mean of 110 and a standard deviation of 10.

a. We have $P(95 < IQ < 115) = P(\frac{95-110}{10} < Z < \frac{115-110}{10}) = \int_{-3/2}^{1/2} \phi(x)dx = \Phi(1/2) - \Phi(-3/2) = .62465$.

b. The 65th percentile for a std normal is .385, so here it is $10*(.385)+110 = 113.85$.

c. For a single person, $P(IQ > 130) = P(Z > \frac{130-110}{10}) = 1 - \Phi(2) = 0.0228$. Now using this as the success probability $p = 0.0228$ for a binomial r.v. X gives $P(X \geq 4) = 5(0.0228)^4(1 - 0.0228) + 0.0228^5 = 1.33 \times 10^{-6}$.

d. Suppose that 500 people are sampled from this distribution. What is the probability of 400 (80%) or more having IQs above 130? We use the normal approximation for the binomial distribution. The expected number of people among

the 500 above IQ 130 is $np = 500 * 0.0228 = 11.4$, the standard deviation is $\sqrt{np(1-p)} = 3.34$. Thus we want to know $P(Z > \frac{400-11.4}{3.34}) = P(Z > 116) = 0$.

e. We have $\overline{X}_{100} \sim N(110, \frac{10}{\sqrt{100}})$ This means the variance of $\overline{X}_{100}$ is one and thus

$$P(\overline{X}_{100} > 112.5) = P(Z > 112.5 - 110) = 0.0062\,.$$

Problem 4. a. Using `x<-rexp(1000)` gives a mean

```
  > mean(x)
[1] 0.9789478
```

and sample variance

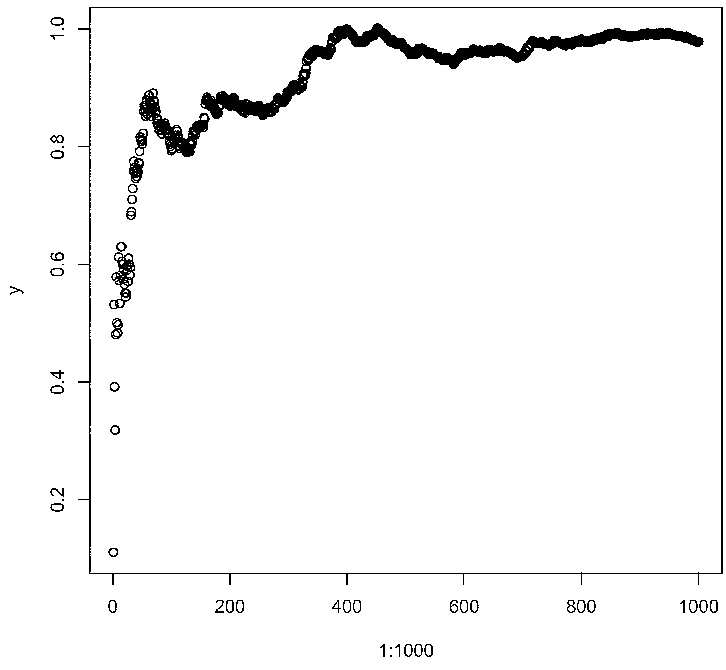
```
> var(x)
[1] 0.9337963
```

Both of these numbers should be close to one (their expected value) because of the law of large numbers.

b. We use the code

```
y <- cumsum(x) / (1 :  length(x))
```

to create a vector of the sequential sample means and `plot(1:1000,y)` to obtain



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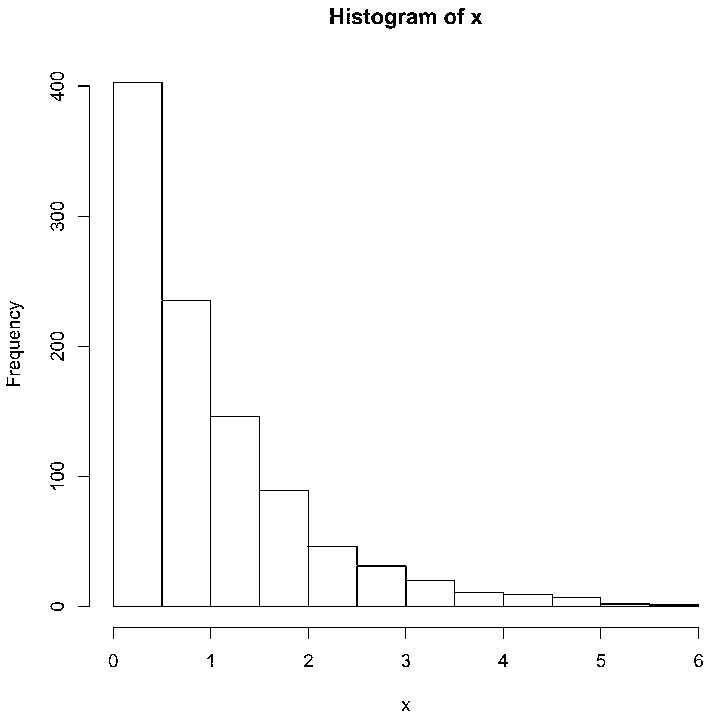
c. The histogram



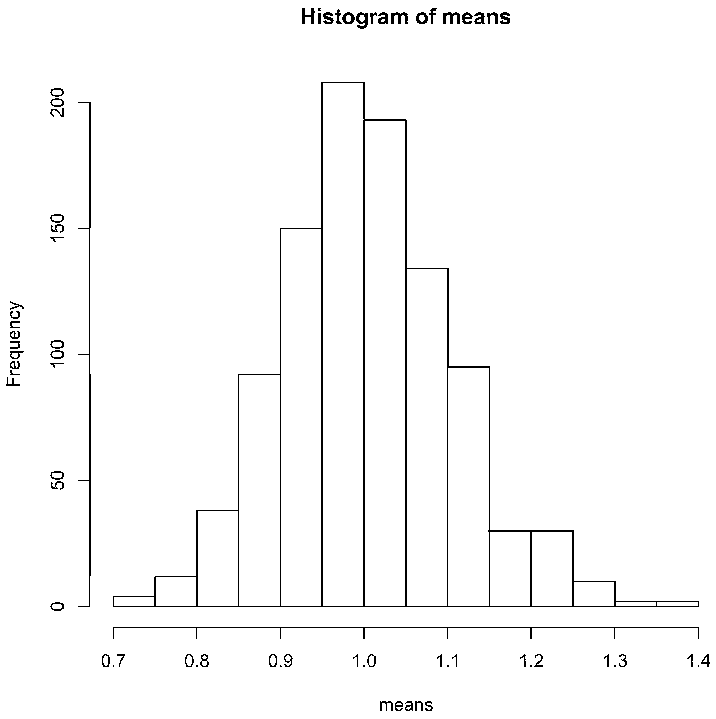
looks roughly like an exponential density because of the law of large numbers (relative frequencies of occurrences for each "bin" tend to probabilities).

d. We use

```
> x<-rexp(1000*100)
> m<-matrix(x,1000,100)
> means<-apply(m,1,mean)
> hist(means)
```

The sample means should have an average close to $x = 1$, and a standard deviation (std error) of $1/\sqrt{100} = \frac{1}{10}$.

e. The histogram looks like that of a normally distributed random variable. This is a consequence of the central limit theorem. (Note that R has already rescaled the axis.)


Histogram of means
Frequency
0
50
100
150
200
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4
means