Assignment 8, due Thursday, November 14, 10am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let \( f(x) = \frac{x}{2} + \frac{1}{x} \). Use some basic calculus to show that \( f \) maps \([1, 2] \) into \([1, 2] \), and use the mean value theorem to show that it is a contraction mapping. What is the value of the unique fixed point \( x_0 \)? If you choose \( x_1 = \frac{3}{2} \) as your starting value, estimate \( |x_0 - x_n| \) for \( n \in \mathbb{N} \).

Problem 2

Let \( f(x) = x^2 - 5 \). Show that \( f \) has a root \( x_0 \) somewhere in the interval \([2, 3] \). Calculate Newton’s \( g(x) \) and prove that \( g \) maps \([2, 3] \) into \([2, 3] \), with \( g'(x) \leq \frac{1}{2} \) for \( x \in [2, 3] \). Prove that if we perform Newton’s method with \( x_1 = 2 \), then \( |x_n - x_0| \leq \frac{1}{2^n} \).

Problem 3

Let \( f(x) = x - \cos(x) \) so if \( x_0 \) is a root for \( f \), then \( \cos(x_0) = x_0 \). Compute Newton’s \( g(x) \) and find concrete numbers \( a \) and \( b \) with \( 0 \leq a \leq b \leq 1 \) such that \( g \) maps \([a, b] \) into \([a, b] \) and it is a contraction mapping. How does the Lipschitz constant of \( g \) compare with the one we had in class when we discussed the fixed point for \( \cos x \)?

Problem 4

Let \( a, y_0 \in \mathbb{R} \). Solve the initial value problem \( y'(x) = ay(x), y(0) = y_0 \) on the interval \([0, \frac{1}{2a}] \) with the help of the contraction mapping theorem.

1. First show that \( \Phi \) as defined in class is a contraction mapping on \( C([0, \frac{1}{2a}]) \).

2. Let \( f_1(t) = y_0 \) and define \( f_{n+1} = \Phi(f_n) \) for \( n \in \mathbb{N} \) as discussed in class. Compute \( f_2 \) and \( f_3 \). Can you guess \( f_n \)?