CHAPTER 2

Continuous Functions

Continuity plays an important role for functions on the real line. Intuitively, a function is continuous provided that it sends "close" points to "close" points. When we say "close" we naturally have a notion of distance between points, both in the domain and the range. Thus, we see that continuity is really a property for functions between metric spaces. In this chapter, we define what we mean by continuous functions between metric spaces, then study the properties of continuous functions. We will see that our notions of open, closed and compact sets all play an important role.

2.1. Continuous functions on metric spaces

1.1. Definition. *Let (X, d) and (Y, ρ) be metric spaces and let $p_0 \in X$. We say that a function $f : X \to Y$ is* **continuous at** *p_0 provided that for every $\epsilon > 0$ there is $\delta > 0$ such that whenever $p \in X$ and $d(p_0, P) < \delta$ then $\rho(f(p_0), f(p)) < \epsilon$.*

Note that in this definition the value of δ really depends on the ϵ for this reason some authors write $\delta(\epsilon)$.
Two quick examples.

1.2. Example. *Let $X = Y = \mathbb{R}$ both endowed with the usual metric and let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Then f is not continuous at $p_0 = 0$.

To see that f is not continuous at 0, take $\epsilon = 1/2$. For any $\delta > 0$, the point $p = \delta/2$ satisfies $d(p_0, p) = |p_0 - p| = |0 - \delta/2| = \delta/2$, but $\rho(f(p_0), f(p)) = |f(p_0) - f(p)| = 0 - 1| > \epsilon$. Thus, every possible value of δ fails to meet the critieria.
Note that the domain of the function is *really* important when trying to decide continuity. For this same formula, if we made the domain instead just $X = [-1, 0]$, then f would be continuous at 0, since now the function would be the function $f(x) = 0$, for every $x \in X$.

1.3. Example. *Let $X = Y = \mathbb{R}$ with the usual metric, let $f(x) = x^2$ and let $p_0 = 3$. Then f is continuous at p_0.*

To see this given $\epsilon > 0$, we take $\delta = \min\{1, \epsilon/7\}$. Then when $d(p_0, p) = |3-p| < \delta$ we know that $|3-p| < 1$ and so $2 < p < 4$. Hence, $d(f(p_0), f(p)) = |3^2 - p^2| = |3-p||3+p| < (3+4)|3-p| < 7\delta \leq \epsilon$.
If we wanted to prove that this function was continuous at $p_0 = 5$, then we could take $\delta = \min\{1, \epsilon/11\}$, and the same argument would work.

1.4. Definition. *Let (X, d) and (Y, ρ) be metric spaces. We say that a function $f : X \to Y$ is* **continuous** *provided that f is continuous at every point in X.*

Thus, f is continuous provided that for each $p_0 \in X$ and $\epsilon > 0$ there is $\delta > 0$, such that $p \in X$ and $d(p_0, p) < \delta$ implies that $\rho(f(p_0), f(p)) < \epsilon$. In this case δ depends on both ϵ and the point p_0. For this reason some author's write $\delta(p_0, \epsilon)$.

1.5. Example. *Let $X = Y = \mathbb{R}$ with the usual metric. Then $f(p) = p^2$ is continuous.*

Given $p_0 \in \mathbb{R}$ and $\epsilon > 0$ let $\delta = \min\{1, \frac{\epsilon}{2|p_0|+1}$. Then $d(p_0, p) < \delta$ implies that $|p| < |p_0| + 1$ and hence $\rho(f(p_0), f(p)) = |p_0^2 - p^2| = |p_0 + p||p_0 - p| \leq (|p_0| + |p|)\delta < (2|p_0| + 1)\delta \leq \epsilon$.
In this example, the δ that we picked depended on both p_0 and ϵ. When it can be chosen independent of p_0, the function is called *uniformly continuous*. That is:

1.6. Definition. *Let (X, d) and (Y, ρ) be metric spaces. We say that a function $f : X \to Y$ is* **uniformly continuous** *provided that for every $\epsilon > 0$ there is $\delta > 0$ such that whenever $p, q \in X$ and $d(p, q) < \delta$ then $\rho(f(p), f(q)) < \epsilon$.*

The following is immediate:

1.7. Proposition. *Let (X, d) and (Y, ρ) be metric spaces and let $f : X \to Y$. If f is uniformly continuous, then f is continuous.*

1.8. Example. *Let $X = Y = \mathbb{R}$ with the usual metric and let $f(p) = 5p + 7$, then f is uniformly continuous.*

Given $\epsilon > 0$, let $\delta = \epsilon/5$, then when $d(p, q) = |p - q| < \delta$ we have that $\rho(f(p), f(q)) = |f(p) - f(q)| = 5|p - q| < 5\delta = \epsilon$.

1.9. Example. *Let $X = Y = \mathbb{R}$ with the usual metric and let $f(p) = p^2$. Then f is not uniformly continuous.*

To prove this it will be enough to take $\epsilon = 1$, and show that for this value of ϵ, we can find no corresponding $\delta > 0$. By way of contradiction, suppose that there was a δ such that $|p - q| < \delta$ implied that $|f(p) - f(q)| < 1$. Set $p_n = n$, $q_n = n + \delta/2$. Then $|p_n - q_n| < \delta$, but $|f(p_n) - f(q_n)| = q_n^2 - p_n^2 = n\delta + \delta^2/4 > n\delta > 1$, for n sufficiently large, in fact for $n > \delta^{-1}$.

1.10. Example. *Let $X = [-M, +M]$, $Y = \mathbb{R}$ with usual metrics and let $f : X \to Y$ with $f(p) = p^2$. Then f is uniformly continuous.*

To see this, given $\epsilon > 0$, set $\delta = \frac{\epsilon}{2M}$. then $|p - q| < \delta$ implies that $|p^2 - q^2| = |p + q||p - q| \leq 2M|p - q| < 2M\delta = \epsilon$.
We now look at a characterization of continuity in terms of open and closed sets.
Recall that if $f : X \to Y$ and $S \subseteq Y$, then the *preimage of* S is the subset of X given by

$$f^{-1}(S) = \{x \in X : f(x) \in S\}.$$

It is important to realize that to make this definition, we do not need for the function f to have an inverse function.

1.11. THEOREM. *Let* (X, d) *and* (Y, ρ) *be metric spaces and let* $f : X \to Y$. *Then the following are equivalent:*

(1) f *is continuous,*
(2) *for every open set* $U \subseteq Y$, *the set* $f^{-1}(U)$ *is an open set in* X,
(3) *for every closed set* $C \subseteq Y$, *the set* $f^{-1}(C)$ *is a closed subset of* X.

1.12. PROPOSITION. *Let* (X, d),(Y, ρ) *and* (Z, γ) *be metric spaces and let* $f : X \to Y$ *and* $g : Y \to Z$ *be continuous functions. Then* $g \circ f : X \to Z$ *is continuous.*

Given a function $f : X \to Y$ and $S \subseteq X$, the *image of S* is the subset of Y given by $f(S) = \{f(p) : p \in S\}$.

1.13. PROBLEM. *Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ a function. Prove that f is continuous at p_0 if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ so that the image of $B_d(p_0; \delta)$ is contained in $B_\rho(f(p_0); \epsilon)$.*

2.2. Continuity and Limits

In this section we generalize the concept of limit and establish the connections between limits and continuity. The first result gives a sequential test for continuity.

2.1. THEOREM. *Let (X, d) and (Y, ρ) be metric spaces, let $f : X \to Y$ be a function and let $p_0 \in X$. Then f is continuous at p_0 if and only if whenever $\{p_n\} \subseteq X$ is a sequence that converges to p_0, the sequence $\{f(p_n)\} \subseteq Y$ converges to $f(p_0)$.*

2.2. COROLLARY. *Let (X, d) and (Y, ρ) be metric spaces and let $f : X \to Y$. Then f is continuous if and only if whenever $\{p_n\} \subseteq X$ is a convergent sequence we have* $\lim_n f(p_n) = f(\lim_n p_n)$.

There is another notion of limit, familiar from calculus, that is also related to continuity.

2.3. DEFINITION. *Let (X, d) and (Y, ρ) be metric spaces, let $p_0 \in X$ be a cluster point, let $f : X \backslash \{p_0\} \to Y$ be a function and let $q_0 \in Y$. We write*

$$\lim_{p \to p_0} f(p) = q_0$$

provided that for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $d(p_0, p) < \delta$ and $p \neq p_0$, then $\rho(q_0, f(p)) < \epsilon$.

Note that we need p_0 to be a cluster point to guarantee that the set of p's satisfying $d(p_0, p) < \delta$ and $p \neq p_0$, is non-empty. Often, when using this definition, the function f is actually defined on all of X, in which case we simply ignore its value at p_0.

2.4. THEOREM. *Let (X, d) and (Y, ρ) be metric spaces, let $f : X \to Y$ be a function and let $p_0 \in X$ be a cluster point. Then f is continuous at p_0 if and only if* $\lim_{p \to p_0} f(p) = f(p_0)$.

2.5. LEMMA. *Let* (X, d) *and* (Y, ρ) *be metric spaces and let* $f : X \to Y$ *be any function. If* $p_0 \in X$ *is not a cluster point, then* f *is continuous at* p_0.

2.6. Corollary. *Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if for every cluster point $p_0 \in X$, we have that $\lim_{p \to p_0} f(p) = f(p_0)$.*

2.3. Functions into Euclidean space

Let (X, d) be a metric space. Given functions $f_i : X \to \mathbb{R}$, $i = 1, ..., k$ we can define a function $F : X \to \mathbb{R}^k$ by setting $F(x) = (f_1(x), ..., f_k(x))$. Conversely, any function $F : X \to \mathbb{R}^k$ is readily seen to be of this form. We call the functions $f_1, ..., f_k$ the *component functions* or, more simply, the *components* of F. To avoid possibly confusion, we shall let d_2 denote the Euclidean metric on $\mathbb{R}^k$.

3.1. Lemma. *Let $X = \mathbb{R}^k$ equipped with the Euclidean metric d_2 and $\mathbb{R}$ with the usual metric. The projection onto the ith coordinate, $\pi_i : \mathbb{R}^k \to \mathbb{R}$, $\pi_i(x) = x_i$, is a continuous function*

3.2. THEOREM. *Let* (X, d) *be a metric space and let* $(\mathbb{R}^k, d_2)$ *be Euclidean space. A function* $F : X \to \mathbb{R}^k$ *is continuous at a point* $p_0 \in X$ *if and only if each of its component functions* $f_i : X \to \mathbb{R}$ *is continuous at* p_0 *for every* $i = 1, ..., k$*. The function* F *is continuous if and only if each of its component functions is continuous.*

3.3. THEOREM. *Let (X, d) be a metric space, let $p_0 \in X$, let $\mathbb{R}$ have the usual metric and let $f, g : X \to \mathbb{R}$ be functions. If f and g are both continuous at p_0, then:*

(1) *$f + g$ is continuous at p_0,*
(2) *fg is continuous at p_0,*
(3) *for any constant c, cf is continuous at p_0,*
(4) *if $g(p) \neq 0$ for all p, then $\frac{f}{g}$ is continuous at p_0.*

3.4. COROLLARY. *Let (X, d) be a metric space, let $\mathbb{R}$ have the usual metric and let $f, g : X \to \mathbb{R}$ be functions. If f and g are both continuous, then:*

(1) *$f + g$ is continuous,*
(2) *fg is continuous,*
(3) *for any constant c, cf is continuous,*
(4) *if $g(p) \neq 0$ for all p, then $\frac{f}{g}$ is continuous.*

3.5. COROLLARY. *Let (X, d) be a metric space, let $p_0 \in X$,and let $F, G : X \to \mathbb{R}^k$. If F, G are both continuous at p_0(respectively, both continuous), then*

(1) *$F + G$ is continous at p_0(respectively, continuous),*
(2) *cF is continuous at p_0(respectively, continuous),*
(3) *$F \cdot G$ is continuous at p_0(respectively, continuous).*

2.4. Continuity of Some Elementary Functions

4.1. PROPOSITION. *Every polynomial defines a continuous function on* $\mathbb{R}$.

4.2. PROPOSITION. *Let p, q be polynomials and let $E = \{x \in \mathbb{R} : q(x) \neq 0\}$. Then $r : E \to \mathbb{R}$ defined by $r(x) = p(x)/q(x)$ is continuous.*

We now prove continuity of our favorite trigonometric functions. We will need a few important facts and inequalities:

$$cos(x+y) = cos(x)cos(y) - sin(x)sin(y), sin(x+y) = sin(x)cos(y) + cos(x)sin(y),$$
$$|sin(x)| \leq |x|, |1 - cos(x)| \leq |x|.$$

The first two equalities are the double angle formulas. The two inequalities follow by examining the unit circle and recalling that x in radian measure is the length of the arc.

4.3. PROPOSITION. *The functions, $sin, cos : \mathbb{R} \to \mathbb{R}$ are both uniformly continuous.*

2.5. Continuous Functions and Compact Sets

If we let $C = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ then it is not hard to see that C is a closed subset of $\mathbb{R}^2$. Also we know that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f((x, y)) = x$ is a continuous function, it is the first coordinate function. But the image $f(C) = \{x \in \mathbb{R} : x \neq 0\}$ is not a closed subset of $\mathbb{R}$. Thus, the continuous image of a closed set need not be a closed set. The story is quite different for compact sets.

5.1. THEOREM. *Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$ be a continuous function. If $K \subseteq X$ is compact, then its image $f(K) \subseteq Y$ is compact.*

5.2. COROLLARY. *Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$ be a continuous function. If X is compact, then $f(X)$ is a bounded subset of Y.*

Thus, when X is non-empty, compact and $f : X \to \mathbb{R}$ is continuous, then there exists M such that $|f(x)| \leq M$ for every $x \in X$. So in particular $\sup\{f(x) : x \in X\}$ and $\inf\{f(x) : x \in X\}$ exists. The following shows that not only does the supremum and infimum exist but that they are attained.

5.3. COROLLARY. *Let (X, d) be a non-empty compact metric space and let $f : X \to \mathbb{R}$ be continuous. Then there are points $x_m, x_M \in X$, such that for any $x \in X$, $f(x_m) \leq f(x) \leq f(x_M)$. That is, $f(x_m) = \inf\{f(x) : x \in X\}$ and $f(x_M) = \sup\{f(x) : x \in X\}$.*

The above result gives the proof that whenever $f : [a, b] \to \mathbb{R}$ is continuous, then f attains its maximum and minimum value. First, by Heine-Borel the interval $[a, b]$ is compact, now apply the above result. Note that $(0, 1)$ is a bounded interval, $f(x) = 1/x$ is continuous on this set but is not even bounded.

5.4. THEOREM. *Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$ be a continuous function. If X is compact, then f is uniformly continuous.*

Thus, for example any rational function $r(x) = p(x)/q(x)$, such that $q(x) \neq 0$ on the set $[a, b]$ will be uniformly continuous.

2.6. Connected Sets and the Intermediate Value Theorem

We will see in this section that the intermediate value theorem from calculus is really a consequence of the fact that an interval of real numbers is a connected set. First we need to define this concept.

6.1. DEFINITION. *A metric space (X, d) is* **connected** *if the only subsets of X that are both open and closed are X and the empty set. A subset S of X is called* **connected** *provided that the subspace (S, d) is a connected metric space. If S is not connected then we say that S is* **disconnected** *or* **separated.**

6.2. PROPOSITION. *A metric space (X, d) is disconnected if and only if X can be written as a union of two disjoint, non-empty open sets.*

6.3. EXAMPLE. *If* (X, d) *is a discrete metric space with two or more points, then* X *is disconnected since* $X = \{p_0\} \cup \{p_0\}^c$ *expresses* X *as a disjoint union of two non-empty open sets.*

We now come to perhaps the most important example of a connected space. By an *interval* in $\mathbb{R}$ we mean either an open interval, closed interval, or half-open interval. The endpoints can be either an actual number or $+\infty$ or $-\infty$.

6.4. THEOREM. *Let* $I \subseteq \mathbb{R}$ *be an interval or all of* $\mathbb{R}$. *Then* I *is a connected set.*

Now we come to a general version of the Intermediate Value Theorem.

6.5. Theorem (Intermediate Value Theorem for Metric Spaces). *Let* (X, d) *be a connected metric space and let* $f : X \to \mathbb{R}$ *be a continuous function. If* $x_0, x_1 \in X$ *with* $f(x_0) < L < f(x_1)$*, then there is* $x_2 \in X$ *with* $f(x_2) = L$*.*

6.6. COROLLARY (Intermediate Value Theorem). *Let $I \subseteq \mathbb{R}$ be an interval or the whole real line and let $f : I \to \mathbb{R}$ be continuous. If $x_0, x_1 \in I$ and $f(x_0) < L < f(x_1)$, then there is x_2 between x_0 and x_1 with $f(x_2) = L$.*

6.7. THEOREM. *Let (X, d) and (Y, ρ) be metric spaces and let $f : X \to Y$ be continuous. If X is connected, then $f(X) \subseteq Y$ is a connected subset.*

6.8. DEFINITION. *A metric space* (X, d) *is called* **pathwise connected** *provided that for any two points,* $a, b \in X$ *there exists a continuous function* $f : [0, 1] \to X$ *such that* $f(0) = a$ *and* $f(1) = b$.

Intuitively, a space is pathwise connected if and only if you can draw a "curve" between any two points with no breaks in the curve.

6.9. EXAMPLE. *The subset of the plane defined by*

$$X = \{(x, sin(\frac{1}{x})) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq +1\}$$

is a space that is connected, but not pathwise connected.