Assignment 1, due Thursday, September 3, 2:30pm

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 0

Prove that if a sequence \(\{p_1, p_2, \ldots\}\) in \(\mathbb{R}\) converges to \(\lim_{n \to \infty} p_n = p\), then it is bounded, that is, there exists a \(L \geq 0\) such that for all \(n \in \mathbb{N}\), \(|p_n| \leq L\). Hint: Start with: Convergence of \(\{p_n\}_{n \in \mathbb{N}}\) means that for every \(\varepsilon > 0\), \(\ldots\) Now choose any \(\varepsilon > 0\), then split the set \(\mathbb{N}\) into two subsets and show the boundedness on each of those subsets.

Problem 1

Suppose \(x\) and \(y\) are unit vectors in \(\mathbb{R}^n\). Show that if \(\left\|\frac{1}{2}(x + y)\right\| = 1\), then \(x = y\).

Problem 2

Let \(x_0 = (a_0, b_0)\) in \(\mathbb{R}^2\), with \(0 < a_0 < b_0\), and define inductively for each \(n \in \mathbb{N}\),
\[
(a_{n+1}, b_{n+1}) = (\sqrt{a_n b_n}, (a_n + b_n)/2).
\]

a. Show, using induction, that for each \(n \in \mathbb{N}\), \(0 < a_n < a_{n+1} < b_{n+1} < b_n\).

b. Estimate \(b_{n+1} - a_{n+1}\) in terms of \(b_n - a_n\).

c. Show that there is \(c \in \mathbb{R}\) such that \(\lim_{n \to \infty} (a_n, b_n) = (c, c)\).

Problem 3

Show that a subset \(S\) of \(\mathbb{R}^n\) is complete if and only if it is closed.

Problem 4

Let \(B_r = \{x \in \mathbb{R}^n : \|x\| < r\}\). Show that a set \(S \subset \mathbb{R}^n\) has no cluster point if and only if \(S \cap B_r\) is a finite set for each \(r > 0\).