Assignment 3, due Thursday, September 17, 2:30pm

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class or in the section on compactness and extreme values in support of your reasoning.

Problem 1

Show that the Cantor set has empty interior. Hint: The construction of the Cantor set can be described in the following way: \( C_0 = [0, 1], \ C_1 = [0, 1] \setminus (1/3, 2/3), \ C_2 = C_1 \setminus ((1/9, 2/9) \cup (4/9, 5/9) \cup (7/9, 8/9)) \) etc., so at each step open intervals are removed. Note that \((4/9, 5/9)\) appears only for notational convenience.

Problem 2

Let \( f \) be defined on \( \mathbb{R} \) by \( f(x) = x \) if \( x \in \mathbb{Q} \) and \( f(x) = 0 \) otherwise. Show that \( f \) is continuous at \( 0 \) but not continuous at any \( a \neq 0 \).

Problem 3

Let \( m : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( m(x, y) = \max\{x, y\} \).

a. Show that \( m \) is continuous.

b. Hence, show that for two real-valued continuous functions \( f \) and \( g \) defined on a set \( S \subseteq \mathbb{R}^n \), the function \( h : S \to \mathbb{R} \) defined by \( h(x) = \max\{f(x), g(x)\} \) is continuous.

Problem 4

Let \( A \) be a compact subset of \( \mathbb{R}^n \). Show that for any \( x \in \mathbb{R}^n \), there is \( a \in A \) which is closest to \( x \) among the points in \( A \), so for any \( y \in A \), \( \|y - x\| \geq \|a - x\| \). Hint: Fix \( x \) and introduce a useful function on \( A \) which you show to be continuous, then quote a result from class.

Problem 5

Assume a real-valued function \( f \) is continuous on \( \mathbb{R}^n \) and satisfies \( f(x) \geq 0 \) for all \( x \in \mathbb{R}^n \) as well as \( \lim_{\|x\| \to \infty} f(x) = 0 \), i.e. for each \( \varepsilon > 0 \) there is \( R > 0 \) such that \( f(x) < \varepsilon \) for all \( x \) with \( \|x\| > R \). Show that \( f \) attains its maximum value.