Department of Mathematics University of Houston

MATH 4331
Introduction to Real Analysis
Fall 2015

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 8, due Thursday, November 5, 2:30pm

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let w be a real-valued, strictly positive continuous function on the interval $[a, b]$. Show that for $f, g \in C([a, b])$, $\langle f, g\rangle_w = \int_a^b f(x)g(x)w(x)dx$ defines an inner product on $C([a, b])$.

Problem 2

Let c_0 be the vector space of all convergent sequences $x = \{x_n\}_{n=1}^\infty$ in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = 0$. Let $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.

a. Show that for $x \in c_0$ there is $k \in \mathbb{N}$ such that $|x_k| = \|x\|_\infty$.

b. Show that $\|\cdot\|_\infty$ defines a norm on c_0.

Problem 3

Let c_0 be the normed space from the preceding problem. Consider a Cauchy sequence $\{x_k\}_{k=1}^\infty$ in c_0 whose elements are denoted by $x_k = \{x_{k,n}\}_{n=1}^\infty$.

a. Show that for each fixed $n \in \mathbb{N}$, $\{x_{k,n}\}_{k=1}^\infty$ is Cauchy in $\mathbb{R}$.

b. Deduce that the limit $y_n = \lim_{k\to\infty} x_{k,n}$ defines $y = \{y_n\}_{n=1}^\infty$ with $\|y\|_\infty < \infty$.

c. Show that $\lim_{k\to\infty} \|x_k - y\|_\infty = 0$

d. Conclude that y belongs to c_0 and thus c_0 is complete.

Problem 4

If $f \in C([0, 1])$ and $1 \le r \le s < \infty$, show that $\|f\|_1 \le \|f\|_r \le \|f\|_s \le \|f\|_\infty$. Hint: Use Hölder's inequality with $g(x) = 1$ and exponent $p = s/r$. Hence, show that if $\{f_n\}_{n=1}^\infty$ in $C([0, 1])$ converges uniformly to $f \in C([0, 1])$, then the sequence also converges with respect to $\|\cdot\|_p$ for $1 \le p < \infty$.