MATH 4331/6312: HANDOUT 1

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CONNECTED SETS AND THE INTERMEDIATE VALUE THEOREM

We will see in this section that the intermediate value theorem from calculus is really a consequence of the fact that an interval of real numbers is a connected set. First we need to define this concept.

1. **Definition.** A subset $A$ of $\mathbb{R}^n$ is disconnected if there are two open sets $U, V$ that are disjoint, $U \cap V = \emptyset$, each of them has a non-empty intersection with $A$, and $A \subseteq U \cup V$. Otherwise, we say that $A$ is connected.

2. **Example.** If $A$ is a non-empty set containing a finite number of points in $\mathbb{R}^n$, then $A$ is disconnected.

The most important example of a connected space is an interval in $\mathbb{R}$, which means either an open interval, closed interval, or half-open interval. The limits can be numbers or $+\infty$ or $-\infty$.

3. **Theorem.** Let $I \subseteq \mathbb{R}$ be an interval or all of $\mathbb{R}$, then $I$ is a connected set. Conversely, if $A$ is a connected set in $\mathbb{R}$, then $A$ is an interval.

**Proof.** Suppose $I$ is an interval but not connected, then we would have $I \subseteq U \cup V$ where $U$ and $V$ are both non-empty and open, $U \cap I \neq \emptyset$, $V \cap I \neq \emptyset$, and $U \cap V = \emptyset$. Let $a \in U$ and $b \in V$. Without loss of generality, we can assume that $a < b$ (otherwise just change the names of $U$ and $V$). Let $U_1 = U \cap I \cap [a, b]$, and $V_1 = V \cap I \cap [a, b]$. Then $U_1$ and $V_1$ are disjoint, non-empty sets because $a \in U_1$ and $b \in V_1$. Since $U_1$ is bounded, $c = \sup\{x : x \in U_1\} < \infty$. From $b \in V$, $a < b$, and $V$ being open, we know there is $\delta > 0$ such that $(v - \delta, v) \subset V_1$. Let $\delta$ be the maximal choice that satisfies this inclusion in $V_1$ (this exists because we know $\delta \leq b - a$). We then know $a \leq c \leq b - \delta$. Moreover, by the assumption on $\delta$, $c \not\in V$, otherwise the openness of $V$ would allow us to increase $\delta$.

Since $c$ is not the right limit of $I$, we also know that $c \not\in U$, otherwise by the openness of $U$ and $c < b$, it would not be the supremum of $U_1$. Hence $c \not\in U \cup V$, but this union has $I$ as its subset, so $c \not\in I$. Thus, $I$ is not an interval.

Conversely, assume $A$ is a connected set in $\mathbb{R}$. Take $a, b \in A$ with $a < b$. To show $A$ is an interval, we prove that each $x \in \mathbb{R}$ with $a < x < b$ satisfies $x \in A$. If this is not the case for some $x$, we can define $U = (-\infty, x)$ and $V = (x, \infty)$, then $a \in U \cap A$, $b \in V \cap A$ but $U$ and $V$ are open and disjoint, hence $A$ is disconnected. \qed

Now we come to a general version of the Intermediate Value Theorem.

4. **Theorem (Intermediate Value Theorem in $\mathbb{R}^n$).** Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $S$ a connected set. Given $x, y \in S$ and $L \in \mathbb{R}$ with $f(x) < L < f(y)$, then there is $z \in S$ with $f(z) = L$.

**Proof.** Suppose there is no such $z$. Let $X = f^{-1}((-\infty, L))$ and $Y = f^{-1}((L, \infty))$, then $X$ and $Y$ are disjoint, $S = X \cup Y$ and by the continuity both of these sets are open in $S$. Consequently, from the definition of relative openness, $X = S \setminus \overline{Y}$ and $Y = S \setminus \overline{X}$. Let $U$ and $V$ be open sets in $\mathbb{R}^n$ such that $X = U \cap S$ and $Y = V \cap S$, then $U_1 = U \setminus \overline{V}$ and $V_1 = V \setminus \overline{U}$ are open and disjoint and we retain the intersection property $X = U \cap S \setminus \overline{V} = U_1 \cap S$ and $Y = V \cap S \setminus \overline{U} = V_1 \cap S$. Thus, $S$ is a subset of two open disjoint sets $U_1$ and $V_1$ in $\mathbb{R}^n$, with a non-empty intersection with each of them, which means it is disconnected, contradicting our assumption. Hence, $L$ is in the range of $f$. \qed
We conclude the usual Intermediate Value Theorem by specializing to \( n = 1 \) and \( S = I \) with \( I \) an interval.

5. **Corollary** (Intermediate Value Theorem). Let \( I \subseteq \mathbb{R} \) be an interval or the whole real line and let \( f : I \to \mathbb{R} \) be continuous. If \( x_0, x_1 \in I, \ L \in \mathbb{R} \) and \( f(x_0) < L < f(x_1) \), then there is \( x_2 \) between \( x_0 \) and \( x_1 \) with \( f(x_2) = L \).

The same type of proof gives another insight that relates continuity and connectedness.

6. **Theorem.** Let \( f : S \subseteq \mathbb{R}^m \to \mathbb{R}^n \) be continuous. If \( S \) is connected, then \( f(S) \) is also connected.

**Proof.** Let \( A = f(S) \). Assume \( A \) is disconnected, then there exist \( U, V \) open disjoint, each having a non-empty intersection with \( A \) and \( A \subseteq U \cup V \). By the continuity of \( f \), \( f^{-1}(U) \) and \( f^{-1}(V) \) are open in \( S \), and satisfy that they are disjoint, have a non-empty intersection with \( S \), and give \( S = f^{-1}(U) \cup f^{-1}(V) \). Using the same argument as in the preceding theorem, \( S \) is disconnected, contradicting our assumption.

Again specializing to \( \mathbb{R} \), we obtain the following consequence.

7. **Corollary.** Let \( I \subseteq \mathbb{R} \) be an interval. If \( f : I \to \mathbb{R} \) is continuous, then \( f(I) \) is an interval.

We recall the informal statement that "a real-valued function is continuous if we can draw its graph without lifting the pen." To make this statement precise, we need a stronger form of connectedness.

8. **Definition.** A set \( A \subseteq \mathbb{R}^n \) is called path connected provided that for any two points, \( a, b \in A \) there exists a continuous function \( f : [0, 1] \to A \) such that \( f(0) = a \) and \( f(1) = b \).

Intuitively, a space is pathwise connected if and only if you can draw a curve between any two points with no gaps in the curve.

9. **Theorem.** If \( A \subseteq \mathbb{R}^n \) is path connected, then it is connected.

**Proof.** Suppose \( A \) is path connected, but not connected. Then there are open sets \( U, V \) with \( A \subseteq U \cup V \), \( U \cap V = \emptyset \) and there are \( a \in A \cap U, b \in A \cap V \). By path connectedness, there is a continuous function \( f : [0, 1] \to A \) with \( f(0) = a \) and \( f(1) = b \). Extending \( f \) to all of \( \mathbb{R} \) by \( f(x) = a \) if \( x < 0 \) and \( f(x) = b \) if \( x > 1 \) retains the continuity. This then gives open disjoint sets \( f^{-1}(U) \) and \( f^{-1}(V) \) whose union contains \([0, 1]\) as a subset and \( 0 \in f^{-1}(U), 1 \in f^{-1}(V) \), so \([0, 1]\) is disconnected, which is a contradiction. Hence, \( A \) is connected.

10. **Definition.** Let \( g : [a, b] \to \mathbb{R} \). By the graph of \( g \) we mean the set \( G = \{(x, g(x)) : a \leq x \leq b\} \subseteq \mathbb{R}^2 \).

11. **Theorem.** Let \( g : [a, b] \to \mathbb{R} \). The function \( g \) is continuous if and only if the graph of \( g \) is a path connected subset of \( \mathbb{R}^2 \).