Practice Exam 2 – Math 4331/6312
November, 2015

First name: ______________ Last name: ______________ Last 4 digits student ID: ________

1 True-False Problems

Put T beside each statement that is true, F beside each statement that is false.

1. If f is bounded and Riemann integrable on $[a, b]$, then $f^2, f^2(x) = (f(x))^2$ is Riemann integrable on $[a, b]$.
2. If f is bounded and Riemann integrable on $[a, b]$ and $0 \leq g(x) \leq f(x)$ for each $x \in [a, b]$, then g is Riemann integrable on $[a, b]$.
3. For an interval $[a, b]$, the set of bounded Riemann integrable functions on $[a, b]$ forms a vector space.
4. If f is bounded and integrable on $[a, b]$, then

   $$\lim_{h \to 0} \frac{1}{h} \int_a^{a+h} f(x)dx$$

   exists.
5. An inner product $\langle x, y \rangle$ is zero if and only if one of the vectors x or y is zero.
6. Equality holds in the triangle inequality on normed vector spaces if and only if the vectors are collinear.
7. A compact subset of a normed vector space is complete.
8. Any bounded sequence in a normed vector space has a convergent subsequence.

In the following problems, you may quote statements from class to simplify your answers. You do not need to give a proof of a statement if it was discussed in class.

2 Problem

Show that if f is bounded and Riemann integrable on $[a, b]$ and $f \geq 0$, then $F(x) = \int_a^x f(t)dt$ defines a non-decreasing function on $[a, b]$.

3 Problem

Let $C^1([0,1])$ denote the vector space of functions on $[0,1]$ that are continuously differentiable. Show that for $f \in C^1([0,1])$,

$$\|f\| = \int_0^1 |f'(x)|dx + |f(0)|$$

defines a norm

4 Problem

Let $\{f_n\}_{n=1}^{\infty}$ be the sequence of functions on $[0,2]$ defined by

$$f_n(x) = \begin{cases} n, & 1/n \leq x \leq 2/n \\ 0, & \text{else} \end{cases}$$

Show that this sequence converges pointwise to the zero function. Show that it does not converge in the L^p-norm for any $1 \leq p < \infty$.

5 Problem

Suppose that f is uniformly continuous on $\mathbb{R}$. Let $f_n(x) = f(x + 1/n)$. Show that f_n converges uniformly to f.

6 Problem

Show that f is Lipschitz continuous on $[0, 1]$ with Lipschitz constant L, then

$$\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{j=1}^{n} f(j/n) \right| \leq L/n\,.$$