Assignment 1, due Thursday, September 7, 10am

Please staple this cover page to your homework. Circle your course number, 4331 or 6312. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 0

Prove that if a sequence \( \{p_n\} \) in \( \mathbb{R} \) converges to \( \lim_{n \to \infty} p_n = p \), then it is bounded, that is, there exists a \( L \geq 0 \) such that for all \( n \in \mathbb{N} \), \( |p_n| \leq L \). Note: Start with: Convergence of \( \{p_n\}_{n \in \mathbb{N}} \) means that for every \( \varepsilon > 0 \), ... Now choose any \( \varepsilon > 0 \), then split the set \( \mathbb{N} \) into two subsets and show the boundedness of the sequence when the index is taken from either one of those two subsets.

Problem 1

Show that the sequence \( \{x_1, x_2, \ldots\} \), defined by \( x_1 = 1 \) and \( x_{k+1} = \frac{1}{2} \left( x_k + \frac{3}{x_k} \right) \), converges to \( \sqrt{3} \). Hint: Assume we already know \( 1 \leq \sqrt{3} \leq 2 \). Re-express the relationship between \( x_k \) and \( x_{k+1} \) as

\[
\frac{x_{k+1} - \sqrt{3}}{x_{k+1} + \sqrt{3}} = \left( \frac{x_k - \sqrt{3}}{x_k + \sqrt{3}} \right)^2.
\]

Problem 2

Suppose \( x \) and \( y \) are unit vectors in \( \mathbb{R}^n \). Show that if \( \| \frac{1}{2}(x + y) \| = 1 \), then \( x = y \).

Problem 3

Let \( x_0 = (a_0, b_0) \) in \( \mathbb{R}^2 \), with \( 0 < a_0 < b_0 \), and define inductively for each \( n \in \mathbb{N} \), \( (a_{n+1}, b_{n+1}) = (\sqrt{a_n b_n}, (a_n + b_n)/2) \).

a. Show, using induction, that for each \( n \in \mathbb{N} \), \( 0 < a_n < a_{n+1} < b_{n+1} < b_n \).

b. Estimate \( b_{n+1} - a_{n+1} \) in terms of \( b_n - a_n \).

c. Show that there is \( c \in \mathbb{R} \) such that \( \lim_{n \to \infty} (a_n, b_n) = (c, c) \).

Problem 4

Show that if a set \( A \) in \( \mathbb{R}^n \) is closed, then it is complete, meaning any Cauchy sequence \( \{p_1, p_2, \ldots\} \) in \( A \) converges and has a limit in \( A \).