Math 4331/6312

Assignment 1

0. Let $\varepsilon = 1$. By convergence of $\{p_k\}_{k=1}^{\infty}$ in $\mathbb{R}$, there is $N \in \mathbb{N}$ s.th. for all $k \geq N$, we have $|p_k - p| < \varepsilon$, so by the triangle inequality $|p_k| < |p| + \varepsilon$.

This gives $\sup_{k \geq N} |p_k| \leq |p| + \varepsilon = |p| + 1$.

Let $L = \max\{ |p_k|, |p|+1 : k < N \}$ then $|p_k| \leq L$ for each $k \in \mathbb{N}$.

1. Given $\{x_k\}_{k=1}^{\infty}$ by $x_1 = 1$, $x_{k+1} = \frac{1}{2}\left(x_k + \frac{3}{x_k}\right)$.

We insert this into

$$y_{k+1} \equiv \frac{x_{k+1} - \sqrt{3}}{x_{k+1} + \sqrt{3}} = \frac{\frac{1}{2}\left(x_k + \frac{3}{x_k}\right) - \sqrt{3}}{\frac{1}{2}\left(x_k + \frac{3}{x_k}\right) + \sqrt{3}}$$

$$= \frac{x_k^2 + 3 - 2\sqrt{3}\, x_k}{x_k^2 + 3 + 2\sqrt{3}\, x_k} = \frac{(x_k - \sqrt{3})^2}{(x_k + \sqrt{3})^2}$$

$$= y_k^2$$

The sequence $\{y_k\}_{k=1}^{\infty}$ observes

$$y_1 = \frac{1-\sqrt{3}}{1+\sqrt{3}}, \quad \text{so by } 1 \le \sqrt{3} \le 2,$$

$$\frac{1-2}{1+1} = -\frac{1}{2} \le y_1 \le \frac{0}{1+1} = 0$$

and thus $0 \le y_2 = y_1^2 = \frac{1}{4}$.

Repeating this gives
$$0 \le y_j = y_{j-1}^2 = y_{j-2}^4 = \cdots = y_1^{2^{j-1}}$$

so
$$\lim_{j\to\infty} y_j = \lim_{j\to\infty} y_1^{2^{j-1}} = \lim_{k\to\infty} y_1^k = 0$$

by $|y_1| < 1$. From $y_j \to 0$, we see that by $y_j = \frac{x_j - \sqrt{3}}{x_j + \sqrt{3}}$ gives $x_j \to \sqrt{3}$.

2. If $\|x\| = \|y\| = 1$ for $x, y \in \mathbb{R}^n$

and $\|\frac{1}{2}(x+y)\| = 1$, then

$$\|\tfrac{1}{2}(x+y)\| = \|\tfrac{1}{2}x + \tfrac{1}{2}y\|$$

$$= 1 = \tfrac{1}{2}\|x\| + \tfrac{1}{2}\|y\|$$

$$= \|\tfrac{1}{2}x\| + \|\tfrac{1}{2}y\|.$$

We see equality holds in the triangle ineq. $\|\frac{1}{2}x + \frac{1}{2}y\| \leq \|\frac{1}{2}x\| + \|\frac{1}{2}y\|$,

so by $\|x\| \neq 0$, $y = \lambda x$, $\lambda \geq 0$.

Inserting y in terms of x gives

$$1 = \|y\| = \|\lambda x\| \overset{\lambda \geq 0}{=} \lambda \underbrace{\|x\|}_{1}$$

so $\lambda = 1$, thus $x = y$.

3. a. We first show for $a_n, b_n > 0$, $a_n \neq b_n$

$a_{n+1} < b_{n+1}$. To see this, note

if $x \neq y$, $x, y > 0$,

$$(x-y)^2 > 0$$

so

$$x^2 + y^2 > 2xy$$

or

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 > xy$$

and subsituting $x = \sqrt{a_n}$, $y = \sqrt{b_n}$

gives

$$\frac{1}{2}a_n + \frac{1}{2}b_n > \sqrt{a_n b_n}$$

or

$$b_{n+1} > a_{n+1}.$$

Thus, from $a_0 < b_0$ induction

gives $a_n < b_n$ for all $n \in \mathbb{N}$, Next, by $b_n > a_n$, we have

$$a_{n+1} = (a_n b_n)^{\frac{1}{2}} > (a_n^2)^{\frac{1}{2}} = a_n$$

and

$$b_{n+1} = \frac{1}{2}(a_n + b_n) < \frac{1}{2}(2b_n) = b_n.$$

This shows

$$0 < a_n < a_{n+1} < b_{n+1} < b_n$$

for each $n \in \mathbb{N}$,

b. By the inequalities, we see that
$$b_{n+1} - a_{n+1} < b_{n+1} - a_n$$
$$< b_n - a_n .$$

(Thus, the sequence $\{d_n\}_{n=1}^{\infty}$ with $d_n = b_n - a_n$ is strictly decreasing and positive.)

c. Moreover, $\{a_n\}_{n=1}^{\infty}$ is increasing, $\{b_n\}_{n=1}^{\infty}$ is decreasing and both are bounded (by b_1 and a_1, respectively).

Thus, there is $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$, and $a \leq b$ by $a_n \leq b_n$. Taking limits gives

$$a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$$
$$= \lim_{n\to\infty} \sqrt{a_n b_n} = \sqrt{ab}$$
$$\Rightarrow \quad \sqrt{a} = \sqrt{b} \quad \Rightarrow \quad a = b .$$

4. Given a Cauchy sequence $\{p_k\}_{k=1}^{\infty}$ in $A \subset \mathbb{R}^n$, so $p_k \in A$ for each $k \in \mathbb{N}$.

By completeness of $\mathbb{R}^n$, we know there is $p \in \mathbb{R}^n$ with $p_k \xrightarrow{k \to \infty} p$.

Next, since $p_k \in A$ for each $k \in \mathbb{N}$ and A is closed, the limit point p is also in A, $\lim_{k \to \infty} p_k = p \in A$.

We conclude A is complete.