Assignment 3, due Thursday, September 21, 10am

Please staple this cover page to your homework. Circle your course number, Math 4331 or 6312. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class or in the section on compactness and extreme values in support of your reasoning.

Problem 1

Show that the Cantor set has empty interior. Hint: The construction of the Cantor set can be described in the following way: $C_0 = [0, 1]$, $C_1 = [0, 1] \setminus (1/3, 2/3)$, $C_2 = C_1 \setminus ((1/9, 2/9) \cup (4/9, 5/9) \cup (7/9, 8/9))$ etc., so at each step open intervals are removed. Note that $(4/9, 5/9)$ appears only for notational convenience.

Problem 2

Let $f$ be defined on $\mathbb{R}$ by $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise. Show that $f$ is continuous at 0 but not continuous at any $a \neq 0$.

Problem 3

Let $m : \mathbb{R}^2 \to \mathbb{R}$ be defined by $m(x, y) = \max(x, y)$.

a. Show that $m$ is continuous.

b. Hence, show that for two real-valued continuous functions $f$ and $g$ defined on a set $S \subset \mathbb{R}^n$, the function $h : S \to \mathbb{R}$ defined by $h(x) = \max\{f(x), g(x)\}$ is continuous.

Problem 4

Let $A$ be a compact subset of $\mathbb{R}^n$. Show that for any $x \in \mathbb{R}^n$, there is $a \in A$ which is closest to $x$ among the points in $A$, so for any $y \in A$, $\|y - x\| \geq \|a - x\|$. Hint: Fix $x$ and introduce a useful function on $A$ which you show to be continuous, then quote a result from class.

Problem 5

Assume a real-valued function $f$ is continuous on $\mathbb{R}^n$ and satisfies $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ as well as $\lim_{\|x\| \to \infty} f(x) = 0$, i.e. for each $\epsilon > 0$ there is $R > 0$ such that $f(x) < \epsilon$ for all $x$ with $\|x\| > R$. Show that $f$ attains its maximum value.