Assignment 7, due Thursday, November 2, 10am

Please staple this cover page to your homework. Circle your course number, 4331 or 6312. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class or on the handout on the Fundamental Theorem of Calculus in support of your reasoning.

Problem 1

Let \( a < b < c \) and \( f \) be a bounded function on \( [a, c] \) that is Riemann integrable on \( [a, b] \) and on \( [b, c] \). Show that \( f \) is Riemann integrable on \( [a, c] \) and that

\[
\int_{a}^{c} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx.
\]

Problem 2

Does the value \( \| (x, y) \| = (|x|^{1/2} + |y|^{1/2})^2 \) for \( (x, y) \in \mathbb{R}^2 \) define a norm on \( \mathbb{R}^2 \)? Explain your answer by supporting it with facts.

Problem 3

Let \( (V, \| \cdot \|) \) be a normed vector space. Assuming convergent sequences \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) in \( V \) with limits \( x \) and \( y \) and a sequence \( \{\alpha_n\}_{n=1}^{\infty} \) in \( \mathbb{R} \) with limit \( \alpha \), show that \( \lim_{n \to \infty} (x_n + y_n) = x + y \) and \( \lim_{n \to \infty} \alpha_n x_n = \alpha x \).

Problem 4

Let \( K \) be a compact subset of \( \mathbb{R}^n \) and let \( C(K, \mathbb{R}^m) \) denote the vector space of all continuous functions from \( K \) to \( \mathbb{R}^m \). Show that if we define \( \| f \|_\infty = \sup_{x \in K} \| f(x) \|_2 \) for each \( f \in C(K, \mathbb{R}^m) \), where \( \| f(x) \|_2 \) is the Euclidean norm of \( f(x) \in \mathbb{R}^m \), then \( \| f \|_\infty \) defines a norm on \( C(K, \mathbb{R}^m) \).

Problem 5

Let \( c_0 \) be the vector space of all convergent sequences \( x = \{x_n\}_{n=1}^{\infty} \) in \( \mathbb{R} \) with \( \lim_{n \to \infty} x_n = 0 \). Let \( \| x \|_\infty = \sup_{n \in \mathbb{N}} |x_n| \).

1. Show that for \( x \in c_0 \) there is \( k \in \mathbb{N} \) such that \( |x_k| = \| x \|_\infty \).

2. Show that \( x \mapsto \| x \|_\infty \) defines a norm on \( c_0 \).