Final Practice Exam – Math 4331/6312
December, 2017

First name: _____ Last name: _____ Last 4 digits student ID: _____

# 1 True-False Problems

Put a T in the box next to each statement that is true, an F for each statement that is false.

- [ ] $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$ is a closed set.
- [ ] If $S = [0, 1) \cup (1, 2]$, then $S$ is connected.
- [ ] An open subset of a compact set in $\mathbb{R}^n$ has a compact closure.
- [ ] All normed real vector spaces are complete.
- [ ] If $\mathcal{F} \subseteq C(K)$, a subset of the set of continuous functions on a compact subset $K$ of $\mathbb{R}^n$, has the property that every sequence in $\mathcal{F}$ has a subsequence that converges to a point in $\mathcal{F}$, then $\mathcal{F}$ is closed.

For the remaining true-false problems, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$

- [ ] If $A$ is compact then $f(A)$ is compact.
- [ ] If a $f$ function is monotonic on $[a, b]$, it is Riemann integrable on $[a, b]$.
- [ ] If $(f_n)_{n=1}^{\infty}$ is a sequence of continuous functions in $C(A)$, with $A$ a subset of $\mathbb{R}$, and the sequence converges uniformly on $A$, then the limit is a continuous function on $A$.

**In the following problems, you may quote statements from class to simplify your answers. You do not need to give a proof of a statement if it was discussed in class.**

## 2 Problem

Let $V$ and $W$ be real vector spaces with two norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively.

Show that $Z = \{(x, y) : x \in V, y \in W\}$, equipped with $\|(x, y)\| = \max\{\|x\|_V, \|y\|_W\}$ is a normed vector space.

## 3 Problem

Show that if $\mathbb{R}$ and $\mathbb{R}^2$ are equipped with the usual (Euclidean) norms, and $K_1$ and $K_2$ are two compact subsets of $\mathbb{R}$, then so is

$$K = \{(x, y) \in \mathbb{R}^2 : x \in K_1, y \in K_2\}.$$

## 4 Problem

Let $A$ be a connected subset of $\mathbb{R}^n$. Prove that the closure of $A$ is also connected.

## 5 Problem

Show that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous, then so is $h = g \circ f, h(x) = g(f(x))$.

## 6 Problem

Let $\ell^p$, $1 \leq p < \infty$ be the normed vector space containing each sequence $a = (a_1, a_2, \dots)$ for which $\|a\|_p = (\sum_{j=1}^{\infty} |a_j|^p)^{1/p} < \infty$. Consider the sequence $\{x_k\}_{k=1}^{\infty}$ with $x_k = (x_{k,1}, x_{k,2}, \dots)$ and

$$x_{k,n} = \begin{cases} 1/k, & n \leq 2^k \\ 0, & n > 2^k \end{cases}$$

show that this sequence does not converge with respect to $\|\cdot\|_p$.

## 7 Problem

Prove that the series $s(x) = \sum_{n=0}^{\infty} x^n$ with partial sums $s_N(x) = \sum_{n=0}^{N} x^n$ defines a sequence of functions $(s_N)_{N=0}^{\infty}$ in $C([0, 1/2])$ which converges with respect to the sup-norm, $\|f\|_\infty = \sup_{0 \le x \le 1/2} |f(x)|$.

## 8 Problem

Show that if $f$ is a real-valued function on the interval $[a, b]$ such that

$$u = \inf\{\int_a^b g(x)dx : g \in C([a,b]), g(x) \geq f(x) \text{ for all } x \in [a,b]\}$$

and

$$l = \sup\{\int_a^b h(x)dx : h \in C([a,b]), h(x) \leq f(x) \text{ for all } x \in [a,b]\}$$

satisfy $u = l$, then $f$ is Riemann integrable.

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