Math 4331/6312
Assignment 1

0. Let \( \varepsilon = 1 \), then by convergence of \( \{p_k\}_{k=1}^{\infty} \), there is \( N \in \mathbb{N} \) s.t. for each \( k \geq N \),

\[ |p_k - p| < \varepsilon = 1, \text{ so by triangle ineq.} \]

\[ |p_k| < |p| + 1. \]

(This gives \( \sup_{k \geq N} |p_k| \leq |p| + 1 \))

Let \( L = \max \{ |p_k|, |p| + 1 : k \in \mathbb{N} \} \)

then \( |p_k| < L \) for each \( k \in \mathbb{N} \).

1. From \( x_1 = 1 \) and \( x_{k+1} = \frac{1}{2} \left( x_k + \frac{3}{x_k} \right) \)

we see \( x_k > 0 \) implies \( x_{k+1} > 0 \),

so all \( x_k > 0 \).

Next, let \( y_k = \frac{x_k - \sqrt{3}}{x_k + \sqrt{3}} \), then

\[
y_{k+1} = \frac{x_{k+1} - \sqrt{3}}{x_{k+1} + \sqrt{3}} = \frac{\frac{1}{2} \left( x_k + \frac{3}{x_k} \right) - \sqrt{3}}{\frac{1}{2} \left( x_k + \frac{3}{x_k} \right) + \sqrt{3}}
\]

\[= \frac{x_k^2 + 3 - 2 \sqrt{3} x_k}{x_k^2 + 3 + 2 \sqrt{3} x_k} = \frac{(x_k - \sqrt{3})^2}{(x_k + \sqrt{3})^2} = y_k^2. \]
We have \(|y_1| = \left| \frac{1 - \sqrt{3}}{1 + \sqrt{3}} \right| < \sqrt{3} - 1 < 1\)

so \(y_2 = y_1^2, \ y_3 = y_2^2 = y_1^4\)

and \(y_{k+1} = y_{k-1}^2 = \ldots = y_1^{2^{k-1}}\)

hence \(\lim_{k \to \infty} y_k = \lim_{k \to \infty} y_1^{2^{k-1}} = \lim_{j \to \infty} y_1^j = 0\).

From \(x_k > 0\),

\[0 < |x_k - \sqrt{3}| < \left| \frac{x_k - \sqrt{3}}{x_k + \sqrt{3}} \right| = |y_k|\]

and \(|y_k| \to 0\), we see \(|x_k - \sqrt{3}| \to 0\),

so \(\lim_{k \to \infty} x_k = \sqrt{3}\).
2. If \( \|x\| = \|y\| = 1 \) for \( x, y \in \mathbb{R}^n \)

and \( \|\frac{1}{2}(x+y)\| = 1 \), then

\[
\|\frac{1}{2}(x+y)\| = \|\frac{1}{2}x + \frac{1}{2}y\| = 1 = \frac{1}{2}\|x\| + \frac{1}{2}\|y\| = \|\frac{1}{2}x\| + \|\frac{1}{2}y\|
\]

We see equality holds in the triangle ineq. \( \|\frac{1}{2}x + \frac{1}{2}y\| \leq \|\frac{1}{2}x\| + \|\frac{1}{2}y\| \), so by \( \|x\| \neq 0 \), \( y = \lambda x, \lambda \geq 0 \).

Inserting \( y \) in terms of \( x \) gives

\[
1 = \|y\| = \|\lambda x\| = \lambda \frac{\|x\|}{1} \quad \lambda \geq 0
\]

so \( \lambda = 1 \), thus \( x = y \).
3. a. We first show for $a_n, b_n > 0$, $a_n \neq b_n$ implies $a_{n+1} < b_{n+1}$. To see this, note if $x \neq y$, $x, y > 0$, then 

$$(x - y)^2 > 0$$

so

$$x^2 + y^2 > 2xy$$

or

$$\frac{1}{2} x^2 + \frac{1}{2} y^2 > xy$$

and substituting $x = \sqrt{a_n}$, $y = \sqrt{b_n}$ gives

$$\frac{1}{2} a_n + \frac{1}{2} b_n > \sqrt{a_n b_n}$$

or

$$a_{n+1} > b_{n+1}.$$ 

Thus, from $a_0 < b_0$ by induction gives $a_n < b_n$ for all $n \in \mathbb{N}$. 

Next, by \( b_n > a_n \), we have
\[
a_{n+1} = (a_n b_n)^{\frac{1}{2}} > (a_n^2)^{\frac{1}{2}} = a_n
\]
and
\[
b_{n+1} = \frac{1}{2} (a_n + b_n) < \frac{1}{2} (2b_n) = b_n.
\]
This shows
\[
0 < a_n < a_{n+1} < b_{n+1} < b_n
\]
for each \( n \in \mathbb{N} \),
b. By the inequalities, we see that
\[ b_{n+1} - a_{n+1} < b_{n+1} - a_n \]
\[ < b_n - a_n. \]

(Thus, the sequence \( \{a_n\}_{n=1}^{\infty} \)
with \( d_n = b_n - a_n \) is strictly decreasing and positive.)

c. Moreover, \( \{a_n\}_{n=1}^{\infty} \) is increasing,
\( \{b_n\}_{n=1}^{\infty} \) is decreasing and both are bounded (by \( b_1 \) and \( a_1 \), respectively).

Thus, there is \( a = \lim_{n \to \infty} a_n \) and
\( b = \lim_{n \to \infty} b_n \), and \( a \leq b \) by \( a_n \leq b_n \). Taking limits gives

\[
a = \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1}
\]
\[
= \lim_{n \to \infty} \sqrt{a_nb_n} = \sqrt{ab}
\]
\[
\Rightarrow \sqrt{a} = \sqrt{b} \Rightarrow a = b.
\]
4. Given a Cauchy sequence \( \{p_k\}_{k=1}^{\infty} \) in \( A \subseteq \mathbb{R}^n \), so \( p_k \in A \) for each \( k \in \mathbb{N} \).

By completeness of \( \mathbb{R}^n \), we know there is \( p \in \mathbb{R}^n \) with \( p_k \to p \).

Next, since \( p_k \in A \) for each \( k \in \mathbb{N} \) and \( A \) is closed, the limit point \( p \) is also in \( A \), i.e., \( p_k = p \in A \) as \( k \to \infty \).

We conclude \( A \) is complete.