Math 4331/6312
Assignment 6

1. For the sake of contradiction, assume \( f \) is not monotonic, then there exist \( \{a,b,c\} \subset [0,1] \), say \( a < c < b \), and
   
   i) \( f(a) < f(c) > f(b) \)
   
   or
   
   ii) \( f(a) > f(c) < f(b) \).

Case i): Let \( M = \max \{ f(c), f(b) \} \),

\( m = \min \{ f(a), f(b) \} \), then \( m < M < f(c) \)

where strict ineq. holds by \( f \) being \( 1-1 \).

Using the IVT, there is \( d \) in

interval between \( f^{-1}(m) \) and \( c \) (either \( (a,c) \) or \( (c,b) \) ) s.th. \( f(d) = M \)

but then \( f(d) = f(a) \) or \( f(d) = f(b) \),

Case ii): With \( M, m \) as above,

\( f(c) < m < M \). Again using the IVT,

there is \( d \) in \( (a,c) \) or \( (c,b) \) s.th.

\( f(d) = m \), so \( f(d) = f(a) \) or \( f(d) = f(b) \).
2. By \( f(x) < g(x) \) for \( x \in (a,b) \),
\[ h(x) = g(x) - f(x) \geq 0. \]

From \( f(x_0) = g(x_0), \quad h(x_0) = 0 \), so \( h \) assumes min at \( x_0 \).

By differentiability of \( f, g \), \( h \) is differentiable and hence using Fermat, \( h'(x_0) = 0 \),
so \( f'(x_0) = g'(x_0) \).

3. From differentiability, there are \( \phi(x), \psi(x) \)

5.) \[ \begin{align*}
    f(x) &= f(x_0) + \phi(x)(x-x_0), \\
    g(x) &= g(x_0) + \psi(x)(x-x_0)
\end{align*} \]

and \( \phi(x) \xrightarrow{x \to x_0} f'(x_0), \quad \psi(x) \xrightarrow{x \to x_0} g'(x_0). \)

We see
\[
\lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x-x_0}
\]
\[
= \lim_{x \to x_0} \frac{(f(x_0) + \phi(x)(x-x_0))(g(x_0) + \psi(x)(x-x_0)) - f(x_0)g(x_0)}{x-x_0}
\]
\[
= \lim_{x \to x_0} \frac{\psi(x)g(x_0)(x-x_0) + f(x_0)\psi(x)(x-x_0)}{x-x_0}
\]
\[
= f'(x_0)g(x_0) + f(x_0)g'(x_0). \]
4. Let \( h(x) = g(x) - f(x) \), then

\[ h \text{ is differentiable and } h'(x) = g'(x) - f'(x) = 0. \]

By mean value theorem on \([a, x]\),

\[ h(x) - h(a) = h'(c)(x-a) = 0 \]

so \( h(x) = h(a) = c \).

Consequently,

\[ g(x) = f(x) + h(x) = f(x) + c. \]

5. a) By differentiability of \( f \),

\[ f(x) = f(a) + \psi(x)(x-a) \]

and \( \lim_{x \to a^+} \psi(x) = f'(a) < 0 \), so there is \( c \) s.t. \( \psi(c) < 0 \) and

\[ f(c) = f(a) + \psi(c)(c-a) < f(a), \]

Also, \( f'(b) > 0 \), so there is \( \psi \) s.t.

\[ f(x) = f(b) + \psi(x)(x-b) \]

and \( \lim_{x \to b^-} \psi(x) = f'(b) > 0 \), so \( \psi(d) > 0 \)

\[ f(d) = f(b) + \psi(d)(d-b) < f(b) \]

\[ > 0 < 0 \]
b) From a), the min does not occur at a or b, so it is at \( x_0 \in (a, b) \).

c) By Fermat, using diffability of \( f \),
\[ f'(x_0) = 0. \]