Department of Mathematics University of Houston

MATH 4331/6312

Introduction to Real Analysis
Fall 2019

First name: ______________________ **Last name:** ________________________ **Points:**

Assignment 9, due Thursday, November 7, 8:30am

Please staple this cover page to your homework. Circle your course number, 4331 or 6312. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b) with $\sup_{a<x<b} |f'(x)| \leq M$ for some $M \geq 0$. Show that the function f is of bounded variation on $[a, b]$, so

$$\sup_P \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| < \infty$$

where the sup is over all partitions $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$. Hint: Mean Value Theorem.

Problem 2

Let $f(0) = 0$ and for $x > 0$, $f(x) = x^2 \sin(1/x)$. Decide if f is of bounded variation on the interval $[0, 1]$ and confirm your opinion by giving a proof.

Problem 3

Recall the function f on $[0, 1]$ that assigns to any rational number $q = m/n$ with m and n relatively prime the value $f(m/n) = 1/n$ and any irrational number r the value $f(r) = 0$. Show that this function is Riemann integrable and that $\int_0^1 f(x)dx = 0$. Hint: Recall that for any $\epsilon > 0$, the set $\{x \in [0, 1] : f(x) \geq \epsilon\}$ is finite.

Problem 4

Suppose that f is non-negative and twice differentiable on $\mathbb{R}$, and assume $\sup_{x\in\mathbb{R}} f(x) = A$ and $\sup_{x\in\mathbb{R}} |f''(x)| = C$. Prove that $\sup_{x\in\mathbb{R}} |f'(x)| \leq \sqrt{AC}$. Hint: If $f'(x_0) = b > 0$, show that $f'(x_0 + t) \geq b - C|t|$. Integrate from $x_0 - b/C$ to $x_0 + b/C$.