1 True-False Problems

Put a T in the box beside each statement that is true, an F if the statement is false.

T Every convergent sequence in \( \mathbb{R}^n \) is bounded.

T Every convergent sequence in \( \mathbb{R}^n \) is Cauchy.

T If \( U \) is an open subset and \( C \) is a closed subset of \( \mathbb{R}^n \), then \( U \setminus C \) is an open subset of \( \mathbb{R}^n \).

T The Cantor set is compact.

F If \( S = [0, 1) \cup (1, 2] \), then \( S \) is a connected subset of \( \mathbb{R} \).

F If \( f : \mathbb{R}^m \to \mathbb{R}^n \) is continuous, then \( f^{-1}(U) \) is open in \( \mathbb{R}^m \) for any set \( U \) that is open in \( \mathbb{R}^n \).

F If \( K \subseteq \mathbb{R}^n \) has the property that every convergent sequence in \( K \) is bounded, then \( K \) is compact.

T If \( S = \{(x, y) : 0 \leq y \leq \frac{1}{x}, x > 0 \} \cup \{(0, y) : y \in \mathbb{R} \} \) then \( S \) is a closed set.

After completing this part, hand it in to obtain the remaining portion of the exam.
In the following problems, you may quote results from class to simplify your answers. You do not need to include a proof of a statement if it was discussed in class.

2 Problem

Prove that if a subset $A$ of $\mathbb{R}^n$ has the property that every Cauchy sequence $\{a_k\}_{k=1}^{\infty}$ in $A$ converges and has a limit $a \in A$, then $A$ is a closed set.

We need to show $A$ contains all of its limit points.

Let $\{a_k\}_{k=1}^{\infty}$ be a sequence which converges and each $a_k \in A$, then $\{a_k\}_{k=1}^{\infty}$ is also Cauchy. By assumption, $\lim_{n \to \infty} a_k = a \in A$, so $A$ contains its limit points.
3 Problem

Let $p_k = \left( \frac{1}{k}, \frac{1}{k^2} \right)$, $k \in \mathbb{N}$, define a sequence in $\mathbb{R}^2$. Prove that $\lim_{k \to \infty} p_k = (0, 0)$.

Let $x_k = \frac{1}{k}$, $y_k = \frac{1}{k^2}$, then

$p_k = (x_k, y_k)$ and $x_k \to 0$

$y_k \to 0$

so by coordinate-wise convergence,

$(x_k, y_k) \xrightarrow{k \to \infty} (0, 0)$,

so $p_k \to (0, 0)$. 
4 Problem

Let $f : [a, b] \to \mathbb{R}$, $a < b$, be a continuous function. Recall that the graph of $f$ is the set

$$G = \{(x, f(x)) : a \leq x \leq b\} \subset \mathbb{R}^2.$$

Show that $G$ is a closed set in $\mathbb{R}^2$.

Let $(p_n)_{n=1}^{\infty}$ be a sequence in $G$ with limit $p \in \mathbb{R}^2$.

By $p_n \in G$, $p_n = (x_n, f(x_n))$.

From $p_n \to p$, $x_n \to x$ and $f(x_n) \to y$ with some $x, y \in \mathbb{R}$. By closedness of $[a, b]$, $x \in [a, b]$.

Using continuity of $f$, $\lim_{n \to \infty} f(x_n) = f(x)$, so $y = f(x)$, hence $p \in G$.

Thus, $G$ contains all of its limit points.
5 Problem

Let \( K \subseteq \mathbb{R} \) be a compact set. Explain why there is \( x \in K \) with \( x \geq y \) for any \( y \in K \).

We know \( K \) is closed and bounded. This means, \( K \subseteq [-R,R] = \overline{B}_R(0) \) for some \( R > 0 \).

Thus, \( \sup \{ x \in K \} \leq R < \infty \).

Next, let \( x_k \to \sup \{ x \in K \} \) with \( x_k \in K \) for each \( k \in \mathbb{N} \).

Then there is conv. subseq. \( \{ x_{k_j} \}_{j=1}^{\infty} \) with \( \lim_{j \to \infty} x_{k_j} = x' \in K \). By subseq. propriety, limit is unchanged, so

\[
\sup \{ x \in K \} = K.
\]
6 Problem

(a) Consider a function \( f : S \subseteq \mathbb{R}^m \rightarrow T \subseteq \mathbb{R}^n \). State the definition of Lipschitz continuity for \( f \).

\( f : S \rightarrow T \) is Lipschitz cont. if there is \( L \geq 0 \) such that for each \( x, y \in S \),

\[ \| f(x) - f(y) \| \leq L \| x - y \|. \]

(b) Prove that if \( f : S \subseteq \mathbb{R}^m \rightarrow T \subseteq \mathbb{R}^n \) and \( g : T \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) are both Lipschitz continuous on their domains, then the composition \( h = g \circ f, h(x) = g(f(x)) \) is Lipschitz continuous on \( S \).

If \( f \) is Lipschitz cont. with const. \( L \) and \( g \) is Lipschitz cont. with const. \( K \), then for each \( x, y \in S \),

\[ \| h(x) - h(y) \| = \| g(f(x)) - g(f(y)) \| \leq K \| f(x) - f(y) \| \leq KL \| x - y \|. \]
We conclude $h$ is Lipschitz cont. with constant $K_L$. 
7 Problem

Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. Show that the graph \( G = \{(x, f(x)) : x \in \mathbb{R}\} \) has a complement \( X = \mathbb{R}^2 \setminus G \) that is disconnected.

Let \( U = \{(x, y) : y > f(x)\} \)

and \( V = \{(x, y) : y < f(x)\} \),

then \( U \cap V = \emptyset \),

\( U \cup V = \{(x, y) : f(x) \neq y\} \)

\( = G' \)

and \( (0, f(c) + 1) \in U \),

\( (0, f(c) - 1) \in V \), so \( U \neq \emptyset \), \( V \neq \emptyset \).

Finally, \( U \) is open by homework problem, and \( V = \{(x, y) : -y > -f(x)\} \), so \( V \) is set "above graph of \(-f\)", reflected about \( x\)-axis, hence open as well.
To see this, let $A$ be an open set and \( B = \{ (x,y) : (x,-y) \in A \} \).

Given \( (x,y) \in B \), then \( (x,-y) \in A \), so there is \( r > 0 \) with \( B_r(x,-y) \subset A \) but then

\[
\{ (u,v) : \| (u,v) - (x,-y) \| < r \} \subset A
\]

so

\[
\{ (u,v) : \| (u,-v) - (x,y) \| < r \} \subset B
\]

hence

\[
B_r(x,y) \subset B
\]

and this means \( B \) is open.