1 True-False Problems

Put T beside each statement that is true, F beside each statement that is false.

F If a real-valued function \( f \) is strictly increasing and \( f \) is differentiable on \([a, b]\), then \( f'(x) > 0 \) for each \( x \in (a, b) \).

T If \( f \) is bounded and Riemann integrable on \([a, b]\), then \( f^2, (f(x))^2 \) is Riemann integrable on \([a, b]\).

F If \( f \) is bounded and Riemann integrable on \([a, b]\) and \( 0 \leq g(x) \leq f(x) \) for each \( x \in [a, b] \), then \( g \) is Riemann integrable on \([a, b]\).

T For an interval \([a, b]\), the set of bounded Riemann integrable functions on \([a, b]\) forms a vector space.

F If \( f \) is bounded and Riemann integrable on \([a, b]\), then for \( a < x < b \),

\[
\lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} f(t) \, dt
\]

exists.
In the following problems, you may quote statements from class to simplify your answers. You do not need to give a proof of a statement if it was discussed in class.

2 Problem

Let \( f : [0, 1] \to \mathbb{R} \) be continuous. What are the possible choices for the range \( f([0, 1]) \)? Support your answer by quoting known facts.

**Two cases:**

a) \( f \) **const.** : \( f([0,1]) = \{c\} \),
with some \( c \in \mathbb{R} \)

b) \( f \) **not const.** : \( f([0,1]) = [a,b] \)
with \( a < b \).

This is because \( f([0,1]) \) is compact,
there is \( c, d \in [0,1] \) s.th.
\( f(c) \leq f(x) \leq f(d) \) for \( x \in [0,1] \),
and \( f(c) \leq f(x) \leq f(d) \) for \( x \in [0,1] \),
so \( \{f(c), f(d)\} \subseteq f([0,1]) \subseteq [f(c), f(d)] \).

By connectedness of \( [0,1] \), and
continuity, \( f([0,1]) \) connected, so
\( f([0,1]) = [f(c), f(d)] \).
3 Problem

Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable on \([-1, 1]\) and \( L : \mathbb{R} \to \mathbb{R} \) the linearization of \( f \) about \( x = 0 \), so the graph of \( L \) is a tangent to the graph of \( f \). Show that for each \( x \in [-1, 1] \),

\[
|f(x) - L(x)| \leq C|x|
\]

with \( C = \sup_{-1 \leq y \leq 1} |f'(y) - f'(0)| \).

We have by differentiability of \( f \) and Mean Value Theorem, for \( x \in [-1, 1] \),

\[
f(x) - f(0) = f'(\xi)(x - 0)
\]

with \( \xi \) between \( x \) and \( 0 \).

Hence,

\[
f(x) - f(0) - f'(0)x \equiv f(x) - L(x)
\]

\[
= f'(\xi)x - f'(0)x
\]

so

\[
|f(x) - L(x)| = |f'(\xi) - f'(0)| |x|
\]

\[
\leq \sup_{-1 \leq y \leq 1} |f'(y) - f'(0)| |x|
\]
4 Problem

Show that if $f$ is bounded and Riemann integrable on $[a, b]$ and $f(t) \geq 0$ for each $t \in [a, b]$, then $F(x) = \int_a^b f(t) \, dt$ defines an increasing function on $[a, b]$.

We know for $x, y \in [a, b]$, $y > x$, $f$ is integrable on $[a, x]$, $[a, y]$, and $[x, y]$, and

$$F(y) = \int_a^y f(t) \, dt = \int_a^x f(t) \, dt + \int_x^y f(t) \, dt$$

so

$$F(y) - F(x) = \int_a^y f(t) \, dt - \int_a^x f(t) \, dt$$

$$= \int_x^y f(t) \, dt \geq 0$$

By monotonicity of $\int_a^b$, $F$ is increasing.
5 Problem

Show that if $f$ is Lipschitz continuous on $[0, 1]$ with Lipschitz constant $L$, then

$$\left| \int_0^1 f(x) \, dx - \frac{1}{n} \sum_{j=1}^n f(j/n) \right| \leq L/n.$$

Let $I_j = [x_{j-1}, x_j]$, $x_j = \frac{j}{n}$, then by a previous homework problem, there is $\tilde{x}_j \in I_j$ s.t.

$$f(\tilde{x}_j) = \frac{1}{x_j - x_{j-1}} \int_{x_{j-1}}^{x_j} f(t) \, dt.$$

So

$$f(\tilde{x}_j) (x_j - x_{j-1}) = \int_{x_{j-1}}^{x_j} f(t) \, dt.$$

Summing these contributions

$$\frac{1}{n} \sum_{j=1}^n f(\tilde{x}_j) = \frac{1}{n} \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(t) \, dt = \int_0^1 f(t) \, dt.$$

Now by Lipschitz cont., $|f(c_{\tilde{x}_j}) - f(c_{\tilde{x}_{j-1}})|$

$$\leq |\frac{j}{n} - \frac{j-1}{n}| L \leq L/n,$$

so

$$\left| \frac{1}{n} \sum_{j=1}^n f(\tilde{x}_j) - \int_0^1 f(t) \, dt \right|$$

$$= \left| \frac{1}{n} \sum_{j=1}^n \left( f(c_{\tilde{x}_j}) - f(c_{\tilde{x}_{j-1}}) \right) \right| \leq \frac{1}{n} \sum_{j=1}^n \frac{L}{n} = \frac{L}{n}.$$
6 Problem

(a) If \( f_1 : [0, 2] \rightarrow \mathbb{R} \) is a bounded function such that \( f_1(x) = 0 \) for all \( x \in [0, 2] \) except at one point \( x_1 \in (0, 2) \) where \( f_1(x_1) = 1 \), then show that \( f_1 \) is Riemann integrable on \([0, 2]\).

For any \( \varepsilon > 0 \), let \( \delta = \min \left\{ \frac{\varepsilon}{3}, x_1, 2-x_1 \right\} \), then choose \( P_1 = \{ 0, x_1-\delta, x_1, x_1+\delta, 2 \} \) and we get

\[
U(f_1, P_1) = (0) (x_1-\delta) + (1) \left( x_1 + \delta - (x_1 - \delta) \right) + (0) \left( 2 - (x_1 + \delta) \right)
\]

\[
= 2\delta \leq \frac{2\varepsilon}{3} < \varepsilon.
\]

By \( L(f_1, P_1) \geq 0 \) from \( f_1 \geq 0 \), we have \( U(f_1, P_1) - L(f_1, P_1) < \varepsilon - 0 \), so \( f_1 \) is Riemann int.

(b) Let \( f \) be real-valued and bounded on \([0, 2]\), with the only non-zero values at \( n \) points \( \{ x_1, x_2, x_3, \ldots, x_n \} \subset (0, 2) \), where \( f(x_i) = 1 \) for each \( i \), then explain why \( f \) is Riemann integrable.

Let \( f_i(x) = \begin{cases} 1 & x = x_i \\ 0, \text{ else} \end{cases} \), then each \( f_i \) is Riemann integrable as shown in (a), so then

\[
f(x) = \sum_{i=1}^{n} f_i(x)
\]

defines a Riemann integrable function.
(c) Let \( f : [0, 2] \rightarrow \mathbb{R} \) be the function
\[
f(x) = \begin{cases} 
1, & \text{if } x = 1/n \text{ and } n \in \mathbb{N}, \\
0, & \text{else}.
\end{cases}
\]

Show that it is Riemann integrable on \([0, 2]\).

Next, let \( S_n(x) = \sum_{i=1}^{n} c_i \chi_{[i/n, (i+1)/n]}(x) \) from (b),
then we have by \( P \) integrability
there is \( P_n \) part of \([0, 2]\) s.t.
then \( U(S_n, P_n) < \frac{\varepsilon}{2} \).
Let \( n > \frac{2}{\varepsilon} \), so
\[
\frac{1}{n} < \frac{\varepsilon}{2}, \quad \text{and let } \ P_n' = P_n \cup \{ \frac{1}{n} \},
\]
then \( U(S_n, P_n') < \frac{\varepsilon}{2} \).

Now, for any \( x_i' \in P_n' \) with \( x_i' < \frac{1}{n} \),
we have \( f(x_i') = S_n(x_i') \leq 1 \)
for any \( x \geq \frac{1}{n}, \ f(x) = S_n(x), \)
so
\[
U(f, P_n') - U(f_n, P_n')
\]
\[
= \sum_{x_i' \leq \frac{1}{n}} (f(x_i') - f_n(x_i')) \Delta_j + \sum_{x_i' > \frac{1}{n}} (0) \Delta_j
\]
\[
\leq \frac{\varepsilon}{\sum_{x_i' \leq \frac{1}{n}} \Delta_j} \cdot \frac{1}{n} + 0
\]
\[
= \frac{\varepsilon}{\sum_{x_i' \leq \frac{1}{n}} \Delta_j} \cdot \frac{1}{n} + 0
\]
Here, by $L(\alpha P_n') \geq 0$,

$$u(f, P_n') - L(f, P_n') \leq u(f, P_n')$$

$$= u(f, P_n') - u(f_n, P_n') + u(f_n, P_n')$$

$$\leq \frac{1}{n} + u(f_n, P_n') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
7 Problem

Suppose \( f : \mathbb{R} \to \mathbb{R} \) is bounded and Riemann integrable on \([a, b]\). Prove that there is \( c \in [a, b] \) such that

\[
\int_a^c f(x) \, dx = \int_c^b f(x) \, dx.
\]

Hint: Define \( F(t) = \int_a^t f(x) \, dx \) and \( G(t) = \int_t^b f(x) \, dx \) and use properties of integrals to deduce that there is \( c \in [a, b] \) such that \( F(c) = G(c) \).

We have \( F(a) = 0 \), \( G(b) = \int_a^b f(x) \, dx \) and

\[
F(t) + G(t) = \int_a^t f(x) \, dx + \int_t^b f(x) \, dx = \int_a^b f(x) \, dx = G(b).
\]

If \( F = \text{constant} \), then

\[
\frac{F(a) + G(a)}{2} = G(a),
\]

but

\[
F(a) = F(b) = G(a).
\]

If not, then \( G(a) \neq F(a) = 0 \), and \( F(a) - G(a) = -G(a) \), \( F(b) - G(b) = F(b) = G(a) \) so \( H(t) = F(t) - G(t) \) changes sign, so by continuity of \( H \) and IVT, there is \( c \in [a, b] \) such that \( H(c) = 0 = F(c) = G(c) \).