1 True-False Problems

Put a T in the box next to each statement that is true, an F for each statement that is false.

T \( \{ (x,y) \in \mathbb{R}^2 : xy = 1 \} \) is a closed set.
F If \( S = [0,1) \cup (1,2] \), then \( S \) is connected.
T An open subset of a compact set in \( \mathbb{R}^n \) has a compact closure.
T If \( \mathcal{F} \subseteq C(K) \), a subset of the set of continuous functions on a compact subset \( K \) of \( \mathbb{R}^n \), has the property that every sequence in \( \mathcal{F} \) has a subsequence that converges to a point in \( \mathcal{F} \), then \( \mathcal{F} \) is closed.

For the remaining true-false problems, consider a real-valued function \( f : \mathbb{R} \to \mathbb{R} \) and a subset \( A \subset \mathbb{R} \)

T If \( A \) is compact and \( f \) is continuous, then \( f(A) \) is compact.
T If \( A \) is bounded and \( f \) is continuous, then \( f(A) \) is bounded.
T If a \( f \) function is monotonic on \([a,b]\), it is Riemann integrable on \([a,b]\).
In the following problems, you may quote statements from class to simplify your answers. You do not need to give a proof of a statement if it was discussed in class.

2 Problem

Show that if $\mathbb{R}$ and $\mathbb{R}^2$ are equipped with the usual (Euclidean) norms, and $K_1$ and $K_2$ are two compact subsets of $\mathbb{R}$, then so is

$$K = \{(x, y) \in \mathbb{R}^2 : x \in K_1, y \in K_2\}.$$

By def., any seq. $(x_n)_{n=1}^{\infty}$ in $K_1$ and any seq. $(y_n)_{n=1}^{\infty}$ in $K_2$ have convergent subseq. by $x_n \in K_1$,

Take $(x_n, y_n) \in K = K_1 \times K_2$, then there is $(x_{n_j})_{j=1}^{\infty}$, $x_{n_j} \to x$. Moreover, $(y_{n_j})_{j=1}^{\infty}$ has subseq. $(y_{n_{j_k}})_{k=1}^{\infty}$ with $y_{n_{j_k}} \to y$ from compactness of $K_2$. Hence, $x_{n_{j_k}} \to x$ and $y_{n_{j_k}} \to y$, so

$$(x_{n_{j_k}}, y_{n_{j_k}}) \to (x, y) \in K_1 \times K_2 \subseteq K.$$

We conclude, $K$ is compact.
3 Problem

Let $V$ and $W$ be real vector spaces with two norms $\| \cdot \|_V$ and $\| \cdot \|_W$, respectively.

Show that $Z = \{(x, y) : x \in V, y \in W\}$, equipped with $\|(x, y)\| = \|x\|_V + \|y\|_W$ is a normed vector space.

By $\|x\|_V \geq 0$, $\|y\|_W \geq 0$, $\|(x, y)\| \geq 0$.

Also, if $\|(x, y)\| = \|x\|_V + \|y\|_W = 0$

if and only if $\|x\|_V = \|y\|_W = 0$

$\implies x = y = 0$

$\iff (x, y) = (0, 0)$.

Next, for $\alpha \in \mathbb{R}$, we have

$\|\alpha (x, y)\| = \|\alpha x, \alpha y)\| = \|\alpha x\|_V + \|\alpha y\|_W$

$= |\alpha| \|x\|_V + |\alpha| \|y\|_W$

$= |\alpha| (\|x\|_V + \|y\|_W)$

$= |\alpha| \|(x, y)\|.$

Finally, $\|(x, y) + (v, w)\| = \|(x + v, y + w)\|$

$= \|x + v\|_V + \|y + w\|_W \leq \|x\|_V + \|v\|_V$

$+ \|y\|_W + \|w\|_W$

$= \|(x, v)\| + \|(y, w)\|.$
4 Problem

Let \((V, \| \cdot \|)\) be a normed vector space. Show that a sequence \((x_n)_{n=1}^{\infty}\) is convergent if and only if it is Cauchy and it has a convergent subsequence.

We know if \((x_n)_{n=1}^{\infty}\) is convergent, it is Cauchy. Also, it has itself as convergent subseq.

Next, let \((x_n)_{n=1}^{\infty}\) be Cauchy and have convergent subseq. \((x_{n_j})_{j=1}^{\infty}\). Let \(\varepsilon > 0\), then by Cauchy, there is \(N \in \mathbb{N}\) with,

for \(n, m \geq N\), \(\|x_n - x_m\| < \frac{\varepsilon}{2}\).

Next, from subseq. , take \(J \) s.th. \(x_{n_j} \rightarrow x\), take \(J \) s.th. \(\|x_{n_j} - x\| < \frac{\varepsilon}{2}\) for \(j \geq J\).

Let \(M = \max \{N, n_J\}\), then for each \(n \geq M\),

\[
\|x_n - x\| \leq \|x_n - x_{n_j}\| + \|x_{n_j} - x\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]

so sequence converges.
5 Problem

Let $A$ be the subset of $\mathbb{R}^2$ given by

$$A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\}.$$ 

Show that $A$ is connected.

We show $A$ is path connected, which implies it is connected.

Let $(x_1, y_1), (x_2, y_2) \in A$ and define

$$(x(t), y(t)) = (1-t)(x_1, y_1) + t(x_2, y_2),$$

then

$$(x(0), y(0)) = (x_1, y_1),$$

$$(x(1), y(1)) = (x_2, y_2)$$

and for $t \in [0, 1]$,

$$(1-t)x_1 + tx_2 \geq 0,$$

$$(1-t)y_1 + ty_2 \geq 0.$$ 

Also, by $x_1 + y_1 \leq 1$, $x_2 + y_2 \leq 1$

$$(1-t)x_1 + x_2 + (1-t)y_1 + ty_2$$

$$= (1-t)(x_1+y_1) + t(x_2+y_2)$$

$$\leq 1-t + t \leq 1,$$ so $(x(t), y(t)) \in A$ for $t \in [0, 1]$. 

6 Problem

Let \( n, m \in \mathbb{N} \) and \( f : \mathbb{R}^n \to \mathbb{R}^m \) be continuous. Show that for any subset \( A \subset \mathbb{R}^n \) with closure \( \overline{A} \), \( f(\overline{A}) \subset \overline{f(A)} \).

We show if \( y \in f(\overline{A}) \), then \( y \in \overline{f(A)} \).

If \( y \in f(\overline{A}) \), then there is \( x \in \overline{A} \) with \( y = f(x) \). So, there is \( (x_n)_{n=1}^\infty \), \( x_n \in A \) for each \( n \in \mathbb{N} \) and \( x_n \to x \).

By continuity, \( y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} f(x_n) = f(x) \)

so \( y \in \overline{f(A)} \).
7 Problem

Prove that the function $f : (0,1] \to \mathbb{R}, f(x) = x \sin(1/x)$ is uniformly continuous.

Let $g : [0,1] \to \mathbb{R}, g(x) = \begin{cases} 0, & x = 0 \\ x \sin(1/x), & x > 0 \end{cases}$

then $|g(x)| \leq x$, so $\lim_{x \to 0^+} g(x) = 0 = g(0)$

so $g$ is continuous on $[0,1]$.

By compactness of $[0,1]$, $g$ is uniformly cont. so for any $\varepsilon > 0$, there is $\delta > 0$ s.t. if $x, y \in [0,1]$ satisfy

$|x - y| < \delta$, then $|g(x) - g(y)| < \varepsilon$.

The same holds for $x, y \in (0,1)$.

But for these, $g(x) = f(x)$ and $g(y) = f(y)$, so $f$ is uniformly cont.
8 Problem

If \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and for each \( x \in \mathbb{R} \), \( f(x+1) = f(x) \), then show the derivative satisfies \( f'(x) = f'(x+1) \) as well.

We have for any \( x \in \mathbb{R} \),

\[
\lim_{{h \to 0}} \frac{f(x+1+h) - f(x+1)}{h} = \lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h} = f'(x)
\]
9 Problem

Show that if \( f : [a, b] \to \mathbb{R} \) is uniformly continuous and \( g : \mathbb{R} \to \mathbb{R} \) continuous, then \( h = g \circ f, h(x) = g(f(x)) \) is uniformly continuous on \([a, b]\).

We know compositions of cont. functions are continuous, so \( h = g \circ f \) is cont.

Also, \( h : [a, b] \to \mathbb{R} \) is defined on a compact domain \([a, b]\), so \( h \) is automatically uniformly continuous.
10 Problem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$ and $g(x) = xf(x)$, show that $g$ is differentiable at $x = 0$.

We have by $g(0) = 0$, $f(0) = 0$, and for $h \neq 0$,

$$
\frac{g(h) - g(0)}{h} = \frac{xf(h) - xf(0)}{h} = \frac{x(f(h) - f(0))}{h}
$$

$$
\lim_{h \to 0} \frac{g(h) - g(0)}{h} = f(0)
$$

so $g'(0)$ exists and $g'(0) = f(0)$. 
11 Problem

Consider the space $C([0, 1])$ equipped with the norm

$$\|f\|_1 = \int_0^1 |f(x)| \, dx$$

and the sequence of functions $(f_n)_{n=1}^\infty$ given by

$$f_n(x) = x^n.$$

Does the sequence $(f_n)_{n=1}^\infty$ converge in this normed vector space?

We have

$$\|f_n\| = \int_0^1 x^n \, dx = \frac{1}{n+1} \to 0$$

so

$$\|f_n - 0\| \to 0$$

means $(f_n)_{n=1}^\infty$ converges to the zero function.