MATH 4332
Introduction to Real Analysis
Spring 2016

First name: ____________ **Last name:** ____________ **Points:**

Assignment 7, due Thursday, April 14, 2:30pm

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Show that if f is a continuous function on $[0,1]$ such that $\int_0^1 f(x)x^n dx = 0$ for each integer $n \geq 0$, then $f \equiv 0$. Hint: Recall that if $\{g_n\}$ is a uniformly convergent sequence in $C([0,1])$ with limit g, then $\lim_{n\to\infty} \int_0^1 g_n(x)dx = \int_0^1 g(x)dx$. Use this idea to first prove $\int_0^1 |f(x)|^2 dx = 0$ and then deduce $f \equiv 0$.

Problem 2

1. If $x_0, x_1, \ldots x_n$ are distinct points in $[a,b]$ and $a_0, a_1, \ldots a_n$ are in $\mathbb{R}$, show that there is a unique polynomial p_a of degree at most n such that $p_a(x_j) = a_j$ for each j. Hint: Find polynomials q_j such that $q_j(x_k) = 0$ for each $k \neq j$ and $q_j(x_j) = 1$. You may express them in factorized form.
2. Next, show that there is a constant M such that for all a, $\|p_a\|_\infty \leq M\|a\|_2$.

Problem 3

Suppose that $f \in C([a,b]$, $\epsilon > 0$ and $x_1, x_2, \ldots x_n$ are points in $[a,b]$. Prove that there is a polynomial p such that $p(x_i) = f(x_i)$ and $\|f-p\|_\infty < \epsilon$. Hint: First approximate f closely by some polynomial of sufficiently high degree, then use the result of the previous problem to adjust this approximation.

Problem 4

If $f \in C([-1,1])$ is an even (odd) function, then show that the best approximation among the polynomials of degree n is also even (odd).

Problem 5

Assume $f \in C([a,b])$ is twice continuously differentiable and $f''(x) > 0$ on $[a,b]$. Show that the best linear approximation (a polynomial of degree one) p to f has the slope $p'(x) = (f(b)-f(a))/(b-a)$.