Assignment 5, due Thursday, March 8, 8:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1
Let \((X,d), (Y,\rho)\) and \((Z,\sigma)\) be metric spaces and \(f : X \to Y\) be a contraction with Lipschitz constant \(r < 1\), \(g : Y \to Z\) a contraction with Lipschitz constant \(s < 1\). Prove that the composition \(h = g \circ f : X \to Z\) has Lipschitz constant \(rs\).

Problem 2
Show that \(f(x) = \sin(x)\) is not a contraction on \([-1, 1]\).

Problem 3
Let \(f(x) = \frac{x}{2} + \frac{1}{x}\). Use some basic calculus to show that \(f\) maps \([1, 2]\) into \([1, 2]\), and use the mean value theorem to show that it is a contraction mapping. What is the value of the unique fixed point \(x^*\)? If you choose \(x_1 = \frac{3}{2}\) as your starting value, estimate \(|x^* - x_n|\) for \(n \in \mathbb{N}\).

Problem 4
Let \(f(x) = \frac{x}{2} - 3\). Starting from \(x_1 = 1\), compute the explicit value of \(x_n\) if we let \(x_n = f(x_{n-1})\). Find the limit \(x^* = \lim_{n \to \infty} x_n\) and verify \(f(x^*) = x^*\).