Assignment 6, due Thursday, March 22, 8:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let \( f(x) = x^2 - 5 \). Show that \( f \) has a root \( x^* \) somewhere in the interval \([2, 3] \). Calculate Newton’s \( g(x) \) and prove that \( g \) maps \([2, 3] \) into \([2, 3] \), with \( g'(x) \leq \frac{1}{2} \) for \( x \in [2, 3] \). Prove that if we perform Newton’s method with \( x_1 = 2 \), then \( |x_n - x^*| \leq \frac{1}{2^n} \).

Problem 2

Let \( f(x) = x - \cos(x) \) so if \( x^* \) is a root for \( f \), then \( \cos(x^*) = x^* \). Compute Newton’s \( g(x) \) and find concrete numbers \( a \) and \( b \) with \( 0 \leq a \leq b \leq 1 \) such that \( g \) maps \([a, b] \) into \([a, b] \) and it is a contraction mapping. How does the Lipschitz constant of \( g \) compare with the one we had in class when we discussed the fixed point for \( \cos x \)?

Problem 3

Let \( f(x) = x^3 - 2 \). Explain why \( x^* = 2^{1/3} \) is the unique (real) root of \( f \). Show that \( 1.25 < 2^{1/3} < 1.26 \). Use Newton’s method to compute \( 2^{1/3} \) to eight decimal places (8 correct digits following the decimal point), starting from \( x_1 = 1.25 \). Using a calculator or a software package for computations is encouraged, however only basic arithmetic is allowed.

Problem 4

Let \( h : [a, b] \times \mathbb{R} \to \mathbb{R} \) be \( C^\infty \), that is, all iterated partial derivatives with respect to the two variables are continuous on \([a, b] \times \mathbb{R} \), and \( T \) be defined as before with an initial value \( y_0 \). Show that for any \( f_1 \in C([a, b]), f_n = T^{n-1} f_1 \) has \( n - 1 \) continuous derivatives. Use this to conclude that the unique solution \( f \) to the differential equation \( f'(x) = h(x, f(x)) \) with initial value \( f(a) = y_0 \) is arbitrarily often continuously differentiable on \([a, b] \).

Problem 5

Let \( h_1 \) and \( h_2 \) be real-valued Lipschitz-continuous functions on \([a, b] \times \mathbb{R} \) and \( f_1 \) and \( f_2 \) be solutions of \( f'_1(x) = h_1(x, f_1(x)) \) and \( f'_2(x) = h_2(x, f_2(x)) \) with initial values \( y_0 = f_1(a) = f_2(a) \). Show that if \( h_1(x, y) \leq h_2(x, y) \) for each \((x, y) \in [a, b] \times \mathbb{R} \), then \( f_1(x) \leq f_2(x) \) for each \( x \in [a, b] \).