1 True-False Problems (4 points each)

Put a circle around T beside each statement that is true, and a circle around F beside each statement that is false.

Throughout \((X, d)\) denotes an arbitrary metric space.

T / F For \(r > 0, p \in X\) the set \(B_r(p)\) is never a closed set.

T / F If \(C \subseteq X\) has the property that every sequence has a subsequence that converges to a point in \(C\), then \(C\) is closed.

T / F If \(f : X \to Y\) is a function, \((X, d)\) is the discrete metric space and \((Y, \rho)\) is any metric space then \(f\) is continuous.

T / F If \(K \subseteq X\) has the property that every convergent sequence in \(K\) is bounded, then \(K\) is compact.

T / F If \(C\) is closed and \(U\) is open, then \(C \cup U'\) is closed.

T / F A closed and bounded subset of a metric space is compact.
In the following problems, you may quote statements from class or homework to simplify your answers.

2 Problem

Let $X = [0, 1) \cup (2, 3]$, with metric $d$ given by $d(x, y) = |x - y|$. Prove that $[0, 1)$ is a closed subset of $X$. 
3 Problem

Let \((X, d)\) be a metric space and \(A \subset X\) be totally bounded. Show that the closure \(\overline{A}\) is totally bounded.
4 Problem

Let $\rho$ be the discrete metric on $\mathbb{R}$ and $d$ be the usual metric on $\mathbb{R}$. Show that the function $f(x) = x$ from $(\mathbb{R}, \rho)$ to $(\mathbb{R}, d)$ is continuous and one-to-one but that $f^{-1}$ is not continuous.
5 Problem

Let \((K,d)\) be a compact metric space and \(f_1, f_2\) (uniformly) continuous, real-valued functions on \(K\). Prove that \(f_1f_2 : x \mapsto f_1(x)f_2(x)\) is uniformly continuous on \(K\).
6 Problem

Let \((X, d)\) be a metric space and let \(K_1, K_2, K_3, \ldots\) be a sequence of nonempty compact sets in \(X\) with \(K_1 \supseteq K_2 \supseteq \cdots \supseteq K_j \supseteq K_{j+1}\) for any \(j \geq 2\), then show that \(\cap_{j=1}^{\infty} K_j\) is non-empty. Hint: Consider a sequence \(\{x_n\}_{n=1}^{\infty}\) with \(x_n \in K_n\) for each \(n \in \mathbb{N}\).