1 True-False Problems

Put T beside each statement that is true, F beside each statement that is false.

☐ If \((X, d)\) and \((Y, \rho)\) denote arbitrary metric spaces, \(X\) is complete and \(f : X \to Y\) is continuous and onto, then \(Y\) is complete.

☐ If a function \(f : [-\pi, \pi] \to \mathbb{R}\) is bounded and Riemann integrable, then its Fourier series is uniformly convergent.

For the remaining true-false problems, all spaces are Euclidean spaces, \(f : \mathbb{R} \to \mathbb{R}\) is continuous and \(A \subset \mathbb{R}\)

☐ If \(A\) is compact then \(f(A)\) is compact.

☐ If \(A\) is closed then \(f(A)\) is closed

☐ If \(A\) is bounded then \(f(A)\) is bounded.

☐ If \(f\) has Lipschitz constant 1, then it has a unique fixed point.

In the following problems, you may quote statements from class to simplify your answers. You do not need to give a proof of a statement if it was discussed in class.
2 Problem

Let $(X, d)$ and $(Y, \rho)$ be metric spaces. Show that $Z = \{(x, y) : x \in X, y \in Y\}$, equipped with $\gamma : Z \to \mathbb{R}$ by $\gamma((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + \rho(y_1, y_2)$ is a metric space.
3 Problem

Suppose $X, Y$ are metric spaces and $f, g : X \rightarrow Y$ are continuous functions. Prove that the set $A = \{x \in X : f(x) = g(x)\}$ is closed in $X$. 
4 Problem

Let $\mathbb{R}$ and $\mathbb{R}^2$ be equipped with the usual (Euclidean) metrics. Prove that if $K_1$ and $K_2$ are two compact subsets of $\mathbb{R}$, then the set

$$K = \{(x, y) : x \in K_1, y \in K_2\}$$

is compact in $\mathbb{R}^2$. 
5 Problem

Let \((X, d)\) be a metric space. Prove that a sequence \(\{p_n\}_{n \in \mathbb{N}}\) is convergent if and only if it is Cauchy and it has a convergent subsequence.
6 Problem

Let \( f : [-1, 1] \to \mathbb{R} \) be continuously differentiable and assume there is \( 0 \leq \epsilon < 1/2 \) such that \( |f'(x) - 1| \leq \epsilon \) for all \( x \in [-1, 1] \).

(a) Show that for any \( x, u \) such that \( x, x + u \in [-1, 1] \), \( |f(x + u) - f(x) - u| \leq |u|/2 \).

(b) With the additional assumption \( f(0) = 0 \), show that \( |x - f(x)| \leq |x|/2 \).
(c) Again with the additional assumption $f(0) = 0$, show that for each fixed $y \in [-1/2, 1/2]$, the function $g_y(x) = y + x - f(x)$ is a contraction mapping on $[-1, 1]$. 
(d) With the assumption as in (b) and (c), show that there exists \( h : \mathbb{R} \to \mathbb{R} \) such that 
\[ h(f(x)) = x \] 
for all \( x \in [-1/4, 1/4] \).
7 Problem

Let $f(x) = e^x$. Find the Taylor series of $f$ at $a = 1$ and state its radius of convergence.
8 Problem

Show that if \( f \) is continuous on \([0, 1]\) and

\[
\int_0^1 f(x) \, dx = 0,
\]

then there is a sequence of polynomials \((p_n)_{n=1}^\infty\) such that \(p_n \to f\) uniformly on \([0, 1]\) and for each \(n \in \mathbb{N}\), \( \int_0^1 p_n(x) \, dx = 0 \).
9 Problem

Let $f(x) = x$ on $[-\pi, \pi]$.

1. Compute the Fourier coefficients of the function $f$. 
2. Use the identity between the squared norm of the function and a series involving the Fourier coefficients to compute $\sum_{k=1}^{\infty} \frac{1}{k^2}$. 
10 Problem

Find the closest linear function $p$ to $f(x) = x^3$ on $[-1, 1]$, meaning $p(x) = ax + b$ for $a, b \in \mathbb{R}$ and $\|f - p\|_{\infty}$ is minimal among all such linear functions in $C([-1, 1])$. Hint: Recall the properties of Chebyshev polynomials and the facts $T_0(x) = 1$, $T_1(x) = x$ and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for $n \geq 2$. 