Assignment 1, due Thursday, January 23, 8:30am

Please staple this cover page to your homework. Circle your course number, 4332 or 6313. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Find all intervals on which the sequence of real-valued functions \((f_n)_{n=1}^{\infty}\) on \(\mathbb{R}\) defined by \(f_n(x) = \frac{x^{2n}}{n + x^{2n}}\) converges uniformly. Explain the reasons supporting your answer.

Problem 2

Show that if the sequence of numbers \((a_n)_{n=1}^{\infty}\) satisfies \(\sum_{n=1}^{\infty} |a_n| < \infty\), then the series \(\sum_{n=1}^{\infty} a_n \cos(nx)\) converges uniformly on \([0,2\pi]\). This means, the partial sums

\[ s_N(x) = \sum_{n=1}^{N} a_n \cos(nx) \]

define a sequence of functions \((s_N)_{N=1}^{\infty}\) that converges uniformly on \([0,2\pi]\). Hint: First show that the sequence is Cauchy with respect to \(|| \cdot ||_{\infty}||.

Problem 3

If \(f \in C([0,1])\) and \(1 \leq r \leq s < \infty\), show that \(||f||_1 \leq ||f||_r \leq ||f||_s \leq ||f||_{\infty}\). Hint: Use Hölder’s inequality with \(g(x) = 1\) and exponent \(p = s/r\). Hence, show that if \((f_n)_{n=1}^{\infty}\) in \(C([0,1])\) converges uniformly to \(f \in C([0,1])\), then the sequence also converges with respect to the norm \(|| \cdot ||_p|| for any \(1 \leq p < \infty\).

Problem 4

Prove that the family \(\mathcal{F} = \{f_n : f_n(x) = \sin(nx), n \in \mathbb{N}, x \in [0,\pi]\}\) is not an equicontinuous family in \(C([0,\pi])\).