1. By comparing exponential and algebraic growth, for $|x| > 1$,

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{(x^2)^n}{n + (x^2)^n} = 1$$

$$\frac{1}{(x^2)^n + 1}$$

and if $|x| \leq 1$,

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{(x^2)^n}{n + (x^2)^n}$$

$$\leq \lim_{n \to \infty} \frac{1}{n + 1} = 0$$

So if $f(x) = \lim_{n \to \infty} f_n(x)$

then $f(x) = \begin{cases} 0, & |x| \leq 1 \\ 1, & |x| > 1 \end{cases}$

and if convex, it is uniform on interval
it cannot contain \([3+1] \) or \([3-1] \).

On the other hand, we have for 
\[\varepsilon > 0, \text{ letting } N > e^{\varepsilon^2} - 1, \text{ then for } h > N, \quad \frac{1}{h+1} < \varepsilon, \] 
so

\[\sup_{|x| \leq 1} \frac{(x^2)^h}{h + (x^2)^h} = \frac{1}{1 + h} < \varepsilon\]

and conv. is uniform.

Next, for \(c > 1\) and \([c, \infty)\), we have
for \(x \in [c, \infty)\),

\[\frac{(c^2)^h}{h + (c^2)^h} = f_n(c) \leq f_n(x) \leq 1\]

\[\downarrow_{h \to \infty}\]

1

so convergence is uniform. Same holds for 
\((-\infty, -c)\) by symmetry \(f_n(-x) = f_n(x)\). Finally, 
by disjoint at \(x = \pm 1\) of \(f\), convergence 
cannot be uniform on \((1, \infty)\) or \((-\infty, -1)\),
2. If  \( \sum_{n=1}^{\infty} |a_n| < \infty \), then

\[
\lim_{N \to \infty} \sum_{n=N}^{\infty} |a_n| = 0,
\]

so for any \( \varepsilon > 0 \), there is \( N \in \mathbb{N} \) such that for \( n, m \geq N \), say \( m > n \),

\[
\sum_{j=n}^{m} |a_j| \leq \sum_{j=N}^{\infty} |a_j| < \varepsilon
\]

hence for \( n, m \geq N \),

\[
\| s_m - s_n \|_{\infty} = \sup_{x \in [0, 2\pi]} \left| \sum_{j=1}^{n} a_j \cos(jx) - \sum_{j=1}^{m} a_j \cos(jx) \right| 
\]

\[
\leq \sup_{x \in [0, 2\pi]} \sum_{j=n+1}^{m} |a_j||\cos(jx)| 
\]

\[
\leq 1
\]

\[
= \sum_{j=n+1}^{m} |a_j| \leq \varepsilon
\]

thus \( (s_n)_{n=1}^{\infty} \) is Cauchy in \( C([0, 2\pi]) \).
By completeness of $C([0,2\pi])$, sequence of partial sums converges in $C([0,2\pi])$.

This is equivalent to uniform convergence.

3. Take $p = \frac{s}{r}$, $s \geq r$

$$\|f\|_r = \left( \int_0^1 |f(x)|^r \, dx \right)^{\frac{1}{r}}$$

$$= \left( \int_0^1 |f(x)|^r (1) \, dx \right)^{\frac{1}{r}}$$

$$\leq \left( \frac{\|f\|_r}{\|p\|_p} \right)^{\frac{s}{r}} \left( \frac{\|1\|_q}{\|q\|_q} \right)^{\frac{1}{s}}$$

$$= \left( \int_0^1 |f(x)|^r \, dx \right)^{\frac{1}{p}}$$

$$s=r^p$$

$$= \left( \int_0^1 |f(x)|^s \, dx \right)^{\frac{1}{s}} = \|f\|_s$$

and

$$\|f\|_s = \left( \int_0^1 |f(x)|^s \, dx \right)^{\frac{1}{s}} \leq \|f\|_\infty.$$
4. If \( F \) were equicontinuous, it would be equicontinuous at each \( a \in [0, \pi] \).

However, for each \( \delta > 0 \), there is \( n \) s.t.
\[ \frac{1}{n} < \delta, \quad \text{so} \quad \left| \frac{1}{n} - 0 \right| < \delta \]

and then if \( x_n = \frac{1}{n} \), we have
\[ \left| \sin(n x_n) - \sin(n 0) \right| = \left| \sin \left( \frac{n \pi}{n} \right) \right| \rightarrow 0 \]

so for any \( 0 < \varepsilon < \sin(1) \), there is \( n \in \mathbb{N} \) with
\[ \left| f_n(x_n) - f_n(0) \right| > \varepsilon \]

hence \( F \) is not equicontinuous at \( a = 0 \).