Assignment 3, due Thursday, February 6, 8:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class or in the preceding term in support of your reasoning.

Problem 1
Show that every open set $A$ in a metric space $(X, d)$ is the union of closed sets.

Problem 2
Let $X = C([0, 1])$ be the space of continuous real-valued functions on $[0, 1]$ with the max-metric

$$d_\infty(f, g) = \max\{|f(t) - g(t)| : 0 \leq t \leq 1\}.$$ 

Prove that the set $P = \{f \in C([0, 1]) : f(t) \geq 0 \text{ for all } 0 \leq t \leq 1\}$ is closed.

Problem 3
Let $(X, d)$ be a metric space and $A \subset X$. The closure $\overline{A}$ of the set $A$ is defined as the set of all $p \in X$ for which there is a sequence $(p_n)_{n \in \mathbb{N}}$ with $p_n \in A$ for each $n \in \mathbb{N}$ and $\lim_{n \to \infty} p_n = p$. Show that $\overline{A}$ is closed and show that if $C$ is a closed set with $A \subset C$, then $\overline{A} \subset C$. Informally, this could be stated as “$\overline{A}$ is the smallest closed set containing $A$”.

Problem 4
Let $(X, d)$ be a metric space, $Y \subset X$ and consider the metric space $(Y, d)$. Show that every open set $U$ in $Y$ has the form $U = V \cap Y$ for an open set $V$ in $X$. Hint: Show this first for open balls.