1. Let \((X, d)\) be \(X = (0, \infty)\) with usual metric from \(\mathbb{R}\) and 
\(Y = X\), \(f : X \to Y\), \(f(x) = \frac{1}{x}\), 
then \(f\) is cont., but \((x_n)_{n=1}^{\infty}\) 
given by \(x_n = \frac{1}{n}\) is Cauchy 
and \((f(x_n))_{n=1}^{\infty}\) has \(f(x_n) = \frac{1}{n} \to 0\) 
so it is not Cauchy because unbounded.

2. a. We show \(x \in f(A) \Rightarrow x \in f(\overline{A})\). 
If \(x \in f(A)\), then there is \(z \in A\), 
\(x = f(z)\). Thus, there is \((z_n)_{n=1}^{\infty}\), \(z_n \in A\) 
for each \(n \in \mathbb{N}\) and \(z_n \to z\). 
By continuity, \(f(z_n) \to f(z) = x\), 
so \(x \in \{f(z_n) : n \in \mathbb{N}\} \subset \overline{f(A)}\). 

b. Let \(X = A = (0, 1)\), \(Y = \overline{\mathbb{R}}\)
\[ f(x) = \frac{1}{x}, \quad \text{then} \]
\[ f(\overline{A}) = f(A) = (1, \infty) \]

but
\[ \overline{f(A)} = (1, \infty) = [1, \infty) \]

4. Let \( \delta = \varepsilon^2 \). If \( x, y \geq 0, \ |x - y| < \delta \), \( \text{WLOG, } x \geq y \), let \( x = y + s \), then \( s < \delta \). From \( \sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \)

letting \( a = y, b = s \) we get
\[
\sqrt{x} - \sqrt{y} = \sqrt{y+s} - \sqrt{y} \leq \sqrt{s}
\]
\[
< \sqrt{\delta} = \varepsilon
\]

hence \( f(x) = \sqrt{x} \) is uniformly cont.

3. If \( g \) is inverse function, so \( g(y) = f^{-1}(xg(y)) \)
then for continuity it is sufficient if
\( g^{-1}(C) \) is closed for any closed set \( C \subset X \), however, \( g^{-1}(C) = f(C) \).

Since \( X \) is compact, \( C \subset X \) ad \( C \) closed implies \( C \) compact, so \( f(C) \) is compact/closed. Thus, \( g \) is continuous.
5. Assume $f$ is cont., but not uniformly so.

Thus, there is $\varepsilon > 0$ for which no $\delta > 0$ works uniformly, so there are $x_\delta, y_\delta$ with $d(x_\delta, y_\delta) < \delta$ but $f(f(x_\delta), f(y_\delta)) \geq \varepsilon$.

Consider $\delta_n = \frac{1}{n}$ and correspondingly $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$.

Choose conv. subsequ. by compactness of $X$, $x_{n_j} \to x$.

Then, from $d(x_{n_j}, y_{n_j}) \to 0$,

$y_{n_j} \to x$. However, then

$z_j = \begin{cases} x_{n_j} \ & \text{if } j \text{ odd} \\ y_{n_j/2} \ & \text{if } j \text{ even} \end{cases}$

has property that $z_j \to x$ but $f(f(z_{j+1}), f(z_j)) \geq \varepsilon$

for $j$ even, because then $z_{j+1} = x_{n_j/2}$,

$z_j = y_{n_j/2}$, so $(f(z_j))_{j=1}^\infty$ does not converge, contradicting continuity of $f$.

Thus, $f$ is uniformly cont.