1. Let $T : X \rightarrow C$ be an isometry, 
\[ T(C) = C, \quad C \text{ complete}. \]

If $C$ is compact, then it is totally bounded, so that for any $\varepsilon > 0$, there exists a finite subset \( \{ x_1, x_2, \ldots, x_N \} \subset C \) such that \( C \subset \bigcup_{j=1}^{N} B_{\varepsilon/2}(x_j) \).

By $T(C)$ dense in $C$, for $x_j \in C$, there is $y_j \in T(C)$ so that $d(x_j, y_j) < \varepsilon/2$.

Hence using triangle inequality, $C \subset \bigcup_{j=1}^{N} B_{\varepsilon}(y_j)$.

From $y_j \in T(x_j)$, there is $x_j \in X$ with $y_j = T(x_j)$. Thus, $C \subset \bigcup_{j=1}^{N} B_{\varepsilon}(T(x_j))$.

Taking inverse under $T$,
\[ X = T^{-1}(C) \subset \bigcup_{j=1}^{N} T^{-1}(B_{\varepsilon}(T(x_j))) \]

$B_{\varepsilon}(x_j)$ via isometry $T$.

So $X$ is totally bounded.

Conversely, if $X$ is totally bounded, then so is $T(X)$ by $T$ being an isometry, and by
previous exercise, \( T(X) \) is total bounded.

Thus, \( T(X) = C \) is complete and total bounded, so it is compact.

2. We first show \((C \times D, \sigma)\) is complete.

Given a Cauchy seq., for any \( \varepsilon > 0 \), there is \( N \) s.t. if \( n, m \geq N \),
\[
\sigma((x_n, y_n), (x_m, y_m)) = d(x_n, x_m) + s(y_n, y_m) < \varepsilon
\]
so
\[
d(x_n, x_m) < \varepsilon
\]
and
\[
s(y_n, y_m) < \varepsilon
\]
Thus \((x_n)_n\) is Cauchy in \( C \),
\((y_n)_n\) is Cauchy in \( D \).

By completeness, \( x_n \to x \) and \( y_n \to y \),
so for \( \varepsilon > 0 \), there is \( N_1 \in \mathbb{N} \) s.t. if \( n \geq N_1 \),
\[
d(x_n, x) < \frac{\varepsilon}{2}
\]
and there is \( N_2 \in \mathbb{N} \), if \( n \geq N_2 \),
\[
s(y_n, y) < \frac{\varepsilon}{2}.
\]
Thus, for \( n \geq \max\{N_1, N_2\} \),
\[
\sigma((x_n, y_n), (x, y)) = d(x_n, x) + s(y_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
so \((x_n, y_n) \to (x, y)\). We conclude \( C \times D \) is complete.
It remains to show $X \times Y = C \times D$
when the closure is w.r.t. $\sigma$, then
$T(x,y) = (x,y)$ gives closed totality.
Since $X = C$, $Y = D$, for any $c > 0$,
$c \in C$, $d \in D$, we have $x \in X$, $y \in Y$
with $d(x,c) < \frac{c}{2}$, $d(y,d) < \frac{c}{2}$, so
$\sigma((x,y), (c,d)) = d(x,c) + g(y,d) < \frac{c}{2} + \frac{c}{2} = c$
hence $X \times Y$ is dense in $C \times D$.

3. For $x_1, x_2 \in X$,
$$\sigma(h(x_1), h(x_2)) = \sigma(g(f(x_1)), g(f(x_2)))$$
$$\leq S \sigma(f(x_1), f(x_2))$$
$$\leq SL d(x_1, x_2)$$
so $h$ has Lipschitz const. $LS$. 
4. Given a cont. $g$, then defining
$f : \mathbb{Q} \to \mathbb{R}$ by $f(x) = g(x)$ gives cont.
function. Moreover, $f|_{[a,b]}$ is uniform
cont. because $[a,b] \subseteq \mathbb{R}$ is compact,
and $f|_{[a,b]}$ has domain $[a,b] \cap \mathbb{Q}$.
By uniform continuity, $f|_{[a,b]}$ has
unique extension on $[a,b]$. Since
$g$ extends $f$, this extension must
be $g|_{[a,b]}$. Now taking the union
of domains $[a,b]$, we get a
consistent extension on $\mathbb{R} = \bigcup_{a < b} [a,b]$
This is identical with $g$. 