Assignment 7, due Thursday, April 2, 8:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1
Show that \( f(x) = \sin(x) \) is not a contraction mapping on \([-1, 1]\).

Problem 2
Let \( f(x) = \frac{x}{2} + \frac{1}{x} \). Use some basic calculus to show that \( f \) maps \([1, 2]\) into \([1, 2]\), and use the mean value theorem to show that it is a contraction mapping. What is the value of the unique fixed point \( x^* \)? If you choose \( x_0 = \frac{3}{2} \) as your starting value, estimate \( |x^* - x_n| \) for \( n \in \mathbb{N} \) with the help of the distance bound in the contraction mapping theorem.

Problem 3
Let \( f(x) = x^2 - 5 \). Show that \( f \) has a root \( x^* \) somewhere in the interval \([2, 3]\). Calculate Newton’s \( g(x) \) and prove that \( g \) maps \([2, 3]\) into \([2, 3]\), with \( g'(x) \leq \frac{1}{2} \) for \( x \in [2, 3] \). Prove that if we perform Newton’s method with \( x_0 = 2 \), then \( |x_n - x^*| \leq \frac{1}{2^n} \).

Problem 4
Let \( f(x) = x - \cos(x) \) so if \( x^* \) is a root for \( f \), then \( \cos(x^*) = x^* \). Compute Newton’s \( g(x) \) and find concrete numbers \( a \) and \( b \) with \( 0 \leq a \leq b \leq 1 \) such that \( g \) maps \([a, b]\) into \([a, b]\) and it is a contraction mapping. How does the Lipschitz constant of \( g \) compare with the one we had in class when we discussed the fixed point for \( \cos x \)?

Problem 5
Let \( f(x) = x^3 - 2 \). Explain why \( x^* = 2^{1/3} \) is the unique (real) root of \( f \). Show that \( 1.25 < 2^{1/3} < 1.26 \). Use Newton’s (improved) method to compute \( 2^{1/3} \) to eight decimal places (8 correct digits following the decimal point), starting from \( x_0 = 1.25 \). Using a calculator or a software package for computations is encouraged, however only basic arithmetic is allowed.