1. We have that $\Phi(f)$ is differentiable, so continuous by FTC, and

$$|\Phi(f)(x)| \leq |a| + \int_0^x |f(t)|xe^{-xt} \, dt \leq \|f\|_{\infty}$$

$$\leq |a| + \|f\|_{\infty} (1 - e^{-x^2})$$

$$\leq |a| + \|f\|_{\infty}$$

so $\Phi$ maps $C([0,1])$ to $C([0,1])$.

Moreover,

$$d_{\infty}(\Phi(f), \Phi(g)) = \sup_{x \in [0,1]} \left| \int_0^x f(t)xe^{-xt} \, dt \right|$$

$$- \left| \int_0^x g(t)xe^{-xt} \, dt \right|$$

$$\leq \sup_{x \in [0,1]} \int_0^x |f(t) - g(t)|xe^{-xt} \, dt$$

$$\leq d_{\infty}(f, g)$$

$$= d_{\infty}(f, g)(1 - e^{-x^2})$$

$$\leq 1 - e^{-b^2} < 1$$
so \( \Phi \) is contraction mapping with unique fixed pt.

This solution \( f \) in \( C([0,b]) \) satisfies

for each \( 0 \leq x \leq b \),

\[
 f(x) = a + \int_0^x f(t) te^{-xt} \, dt.
\]

Comparing solutions for \( C([0,b]) \) and \( C([0,b']) \) with \( b' > b \), they are consistent on \( [0,b] \), so each solution extends to \( U [0,b] = R^+ \).

In each interval, the solution is unique, hence so it is on \( [0,\infty) \).

2. Considering

\[
\Phi(f)(x) = 1 + \int_0^x t f(t) \, dt
\]

we see \( h(x,y) = xy \) is Lipschitz cont.

in \( y \) with \( \| h(x,y) - h(x,y') \| \leq \sup_{x \in [0,b]} \| x \| |y-y'| \)

\[
\leq \frac{1}{2} |y-y'|
\]

hence \( \Phi \) is contraction mapping on \( C([0,b]) \)

with Lipschitz const \( r = \frac{1}{2} (\frac{1}{b}-0) = \frac{1}{b} \).
Starting from \( f_0(x) = 1 \),
\[
f_1(x) = 1 + \int_0^x t(x) \, dt
\]
\[
= 1 + \frac{x^2}{2}
\]
so on \([0, \frac{1}{2}J] \),
\[
d_{\infty}(f_0, f_1) = \sup_{x \in [0, \frac{1}{2}J]} |1 + \frac{x^2}{2} - 1| = \frac{1}{8}
\]
we get for solution \( f^* \)
\[
\|f_n - f^*\|_{\infty} \leq \frac{1}{8} \left( \frac{1}{4} \right)^n
\]
\[
= \frac{1}{6} \left( \frac{1}{4} \right)^n
\]

3. Let \( f_0 \in C([a, b]) \), then by \( h \in C^0 \),
\[
f_1(x) = y_0 + \int_a^x h(t, f_0(t)) \, dt
\]
so \( f_1 \) by FTC is cont. differentiable.
Assuming \( f_n \) is \( n \) times cont. differentiable,
then
\[
f_{n+1}(x) = y_0 + \int_a^x h(t, f_n(t)) \, dt
\]
defines \((n+1)\) - times cont. diff. function because

\[ f_{n+1}(x) = h(x, f_n(x)) \]

is \(n\) times differentiable via chain rule and \(f_n\) being \(n\) times cont. and diff. function

This implies, if

\[ f(x) = \int_a^x h(t, f(t)) \, dt + y_0 \]

then setting \(f_0 = f\), we have \(f_1 = f\), \(f_2 = f\), ... so \(f\) is an arbitrary often differentiable function.
4. a) Consider 
\[ \Phi (f)(x) = \int_0^x \left( 1 + (f(t)y)^2 \right) \, dt \]

Letting \( f_y(x) = y \), we get

\[ d_\infty (\Phi(f_y), \Phi(f_y')) = \sup_{x \in [0,b]} \left| \int_0^x (1 + y^2) \, dt + \int_0^x C(1 + (y')^2) \, dt \right| \]

\[ = b \left| y^2 - (y')^2 \right| \]

Setting \( y' = 0 \), \( y = \frac{1}{b} \) gives

\[ d_\infty (\Phi(f_{\frac{1}{b}}), \Phi(f_0)) = \frac{1}{b} \cdot \frac{1}{b} = d_\infty (f_{\frac{1}{b}}, f_0) \]

so \( \Phi \) is not a contraction mapping.
5) Consider \( f \in \overline{B}_{1,0}(0) \subset C([0,b]) \) then

\[
|\Phi(f)(x)| = \left| \int_0^x (1 + (f(t))^2) \, dt \right|
\]

\[
\leq \left| 1 \times (1 + \|f\|_{\infty}) \right| \leq 1
\]

\[
\leq 2|x| \leq 2b
\]

so if \( b < \frac{1}{2} \), then \( \Phi \) maps \( C([0,b]) \) to \( C([0,b]) \).

Moreover, for \( f, g \in C([0,b]) \),

\[
|\Phi(f)(x) - \Phi(g)(x)|
\]

\[
= \left| \int_0^x (1 + (f(t))^2) \, dt - \int_0^x (1 + (g(t))^2) \, dt \right|
\]

\[
= \left| \int_0^x (f(t))^2 - (g(t))^2 \, dt \right|
\]
\[ M = \left| \int_{0}^{x} \left( \frac{|f(t) + g(t)|}{2} - \frac{|f(t) - g(t)|}{2} \right) \, dt \right| \leq 2 \| f - g \|_{C[0,16]} \]

\[ \leq 2 \, b \, \| f - g \|_{C[0,16]} \]

so again if \( b < \frac{1}{2} \), then this is a contraction mapping.

(c. Thus, for any \( b < \frac{1}{2} \), \( y' = 1 + y^2 \)

has a unique sol. to IVP \( y(0) = 0 \).

Since \( \phi \) maps \( B_1(0) \) in \( C([0,16]) \)

to itself, unique solution extends continuously

to \([0, \frac{16}{2}]\).